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On Anholonomic Deformation, Geometry, and Differentiation

by John D. Clayton

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On anholonomic deformation, geometry, and differentiation

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Abstract
In geometrically nonlinear theories of inelasticity of solids, the deformation gradient is typically split multiplicatively into two (or more terms), none of which need be integrable to a motion or displacement field. Such terms, when not integrable, are termed anholonomic, and can be associated with intermediate configuration(s) of a deformed material element. In this work, aspects of tensor calculus associated with anholonomic deformation are analyzed in general curvilinear coordinates. Various linear connection coefficients for intermediate configurations are posited or derived; of particular interest are those mapped coefficients corresponding to the choice of identical basis vectors in multiple configurations. It is shown that torsion and curvature associated with such mapped coefficients do not necessarily vanish, even though torsion and curvature tensors of the original connections vanish by definition in reference or current configurations. Intermediate connection coefficients defined in this way exhibit vanishing covariant derivatives of corresponding metric tensors, but are time dependent even when reference (current) configuration connections are fixed in time at a given material (spatial) location. Formulae are derived for total covariant derivatives of two- and three-point tensors with one or more components referred to the intermediate configuration. It is shown that in intermediate coordinates, neither the divergence of the curl of a vector field nor the curl of the gradient of a scalar field need always vanish. The balance of linear momentum for a hyperelastic–plastic material is examined in the context of curvilinear intermediate coordinates.

Keywords
anholonomic deformation, curvature, differential geometry, finite strain, multiplicative decomposition, torsion

1. Introduction

The notion of a locally relaxed or stress-free intermediate configuration is widely used in geometrically nonlinear (i.e. finite deformation) models of solids. An intermediate state in which deformations of neighboring material elements may be incompatible was proposed for anelastic materials [1]. A multiplicative decomposition of the deformation gradient into elastic and plastic parts was developed for crystalline solids containing continuous distributions of dislocations [2, 3]. The torsion tensor constructed from gradients of elastic or plastic deformations can be associated with the density of dislocations [3–5]. Shortly thereafter, theories of continuous bodies with inhomogeneities, in which various non-integrable deformation mappings are introduced, were developed [6, 7]. Subsequent literature on nonlinear kinematics of crystals with defect distributions is immense; relevant more recent works include [8–14].

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The multiplicative decomposition of the deformation gradient for crystals deforming via dislocation-mediated plasticity has been extended to describe kinematics of a number of other physical phenomena. For example, such decompositions have been used to describe thermal expansion/contraction [15], growth mechanics in biological tissues [16], volume changes associated with point defects [10, 17] and voids [18], deformation twinning [19, 20], and disclination defects [21, 22]. Theories of inhomogeneous bodies have been applied to describe solid and fluid crystals and laminated composite materials [7, 23] in addition to isotropic solids and anisotropic solid crystals. Extensions of the multiplicative decomposition to three (or more) terms have been used to quantify elastic–plastic phenomena at different length scales [24]. Reviews with more comprehensive lists of references are available elsewhere [14, 25]. It should be noted that theories for nonlinear material behavior in the context of defective crystals [8] and biological systems [26] have been developed that do not require explicit use of a multiplicative decomposition.

In the present work, the total deformation gradient $\mathbf{F}$ is decomposed as $\mathbf{F} = \bar{\mathbf{F}}\tilde{\mathbf{F}}$, where $\tilde{\mathbf{F}}$ and $\bar{\mathbf{F}}$ are generic two-point mappings from the reference to intermediate configuration and intermediate to current configuration, respectively. In coordinates, $F^a_{.A} = \partial\varphi^a/\partial X^A$ is the tangent map from the reference to current configuration, where spatial coordinates $x^a = \varphi^a(X, t)$ follow the motion φ that may depend on time t and material particle X. Deformation gradient $\mathbf{F}$ is said to be integrable or 'holonomic' since the differentiable one-to-one mapping $\varphi(X, t)$ exists between referential and spatial positions of material particles. On the other hand, $\tilde{F}^\alpha_{.A}(X, t)$ and $\bar{F}^{-1\alpha}_{\ .a}(x, t)$ need not be integrable functions of X^A and x^a, respectively; in such cases, these mappings are said to be 'anholonomic' [5, 14, 27, 28]. Correspondingly, when $\tilde{\mathbf{F}}$ ($\bar{\mathbf{F}}^{-1}$) is not integrable, coordinates $\tilde{x}^\alpha$ that are differentiable one-to-one functions of X^A (x^a) do not exist, and the intermediate configuration is said to be an anholonomic space.

According to Schouten [27], the study of differential geometry of anholonomic spaces was initiated by Vranceanu [29]. General mathematical identities are derived or listed in [27]; anholonomic coordinate transformations are also mentioned by Ericksen [30, p. 801] in the context of tensor fields in mechanics. Applications of anholonomic geometry to plasticity theory are described by Kondo [5, 31].

In a previous work co-authored by the present author [28], several different metric tensors on the intermediate configuration were discussed. Of particular interest in that work was a metric tensor formed from the scalar product of spatial basis vectors convected to the intermediate configuration using the elastic distortion. This metric, which corresponds to the right Cauchy–Green deformation tensor formed from the elastic deformation gradient, demonstrates a non-vanishing curvature tensor associated with incompatibilty or anholonomicity of the intermediate configuration, and proves useful in formulating scalar energy potentials for crystalline solids with continuous distributions of dislocations [14, 28]. In that work [28], the same notation was used for different sets of anholonomic and convected basis vectors; the present work corrects this ambiguous notation.

The usual choice of basis vectors in the intermediate configuration is a Cartesian system with metric tensor components equivalent to Kronecker's delta symbols. In theories of crystal elasto-plasticity, coincident Cartesian coordinates are almost universally used for every configuration of a material body. Cartesian coordinates prove especially convenient for representing anisotropic elastic and plastic behavior of single crystals, for which elastic moduli and slip system geometry are most easily described using Cartesian frames of reference [14, 25, 32, 33]. Teodosiu [34] suggested an intermediate coordinate system wherein an orthonormal director triad is attached to each locally relaxed material point, but triads at different material points can differ by a finite rotation; the field of such rotations presumably may be discontinuous in material coordinates. In such a representation, the metric tensor also reduces to Kronecker's delta, and it was assumed that connection coefficients associated with covariant differentiation in the (possibly anholonomic) intermediate configuration vanish identically [34].

Of particular interest in the present paper are general curvilinear coordinates in the intermediate configuration. Curvilinear coordinates such as cylindrical or spherical systems are useful for describing bodies whose shapes are naturally extrinsically or intrinsically curved, e.g. cylinders, spheres, and shells of various kinds. Such shapes arise frequently in biological systems. Furthermore, certain crystalline structures may be amenable to description with curvilinear coordinates; for example screw dislocations perpendicular to the basal plane in crystals with hexagonal symmetry (with second-order elastic constants having transverse isotropy) may be described using cylindrical coordinates [35]. In such situations, it is natural for one to use the same system of coordinates in all configurations if the shape of the body remains roughly the same during deformation. For example, if the body remains cylindrical throughout the deformation process, the natural choice of coordinates would be cylindrical coordinates in reference, intermediate, and current configurations. Simo [36] developed a finite elastic–plastic theory wherein the same metric tensor and presumably the same coordinate system, which may generally be curvilinear, are used for both reference and intermediate configurations. The present author [14, p. 91] suggested that two pragmatic choices for intermediate coordinate systems are either this prescription [36] or the prescription of the same, possibly curvilinear, coordinate system in intermediate and spatial configurations.

The present paper supplements, refines, and substantially extends a previous treatment of anholonomic geometry in the context of solid mechanics (in particular, brief sections 2.8 and 3.2.3 of [14]). Choices of coordinate systems (i.e. basis vectors) in the potentially anholonomic intermediate configuration are critically examined. Metric tensors, connection coefficients, torsion, and curvature are derived for each choice of basis.

As shown for what appears to be the first time in this paper, the choice of such mapped intermediate coordinate systems from curvilinear referential or spatial coordinate systems [14, 36] leads to intermediate basis vectors whose derived connection coefficients may be non-symmetric in covariant indices and may have non-vanishing torsion and curvature. It is also shown that torsion and curvature may be non-zero even if the intermediate configuration is holonomic. Upon development of logical definitions for partial and covariant differentiation with respect to possibly anholonomic intermediate coordinates, it is shown that the divergence of the curl of a vector field and the curl of the gradient of a scalar field need not necessarily vanish in the intermediate configuration. Piola's identity for the Jacobian determinant [14, 37, 38] also does not generally apply in the intermediate configuration. Formulae are derived for total covariant derivatives [30, 39] of two- and three-point tensors with one or more components referred to the intermediate configuration. It is shown by example that such formulae are needed when writing the balance of linear momentum for nonlinear hyperelastic–plastic solids in general curvilinear coordinates.

Much of this paper is tutorial in nature (e.g. the content of Section 2 can be found in books on nonlinear continuum mechanics and elasticity in curvilinear coordinates); however, in addition to serving as a useful reference, such content is needed to develop and contrast results in subsequent sections. As mentioned in the preceding paragraph, the present work contains apparently new results pertaining to particular choices of intermediate coordinate systems that are not explicitly evident in other relevant works incorporating direct (i.e. coordinate-free) notation [6, 40].

This article is organized as follows. Definitions and notation for geometry and kinematics of holonomic (i.e. integrable) deformation gradients are given in Section 2. Kinematics and geometry of anholonomic deformation are described in Section 3, including integrability conditions, choices of coordinate systems, rules for partial and covariant differentiation, and important derived identities. The local balance of linear momentum in geometrically nonlinear continuum mechanics is examined in Section 4 in the context of possibly curvilinear intermediate coordinate systems. Conclusions follow in Section 5, while an Appendix contains explicit forms of intermediate connection coefficients and curvature for cylindrical coordinates. Notation of nonlinear continuum mechanics [14, 38, 39] is used. Einstein's summation convention applies over repeated indices. Uppercase Roman font is used for indices corresponding to referential (i.e. material) coordinates, lowercase Roman font for current (i.e., spatial) coordinates, and lowercase Greek font for intermediate (and possibly anholonomic) coordinates.

2. Holonomic deformation

2.1. Configurations, coordinates, and metrics

A material point or particle in reference configuration B_0 is labeled X. The corresponding point in the current or spatial configuration B is labeled x. Time is denoted by t. The motion and its inverse, respectively, are

$$x^a = \varphi^a(X, t) = x^a(X, t), \quad X^A = \Phi^A(x, t) = X^A(x, t). \tag{1}$$

Unless noted otherwise, spatial coordinates x^a and reference coordinates X^A are assumed sufficiently differentiable functions of their arguments. Partial differentiation is written alternatively as follows:

$$\frac{\partial(\cdot)}{\partial X^A} = \partial_A(\cdot) = (\cdot)_{,A}, \quad \frac{\partial(\cdot)}{\partial x^a} = \partial_a(\cdot) = (\cdot)_{,a}. \tag{2}$$

The following identities are used frequently:

$$\partial_A[\partial_B(\cdot)] = \partial_B[\partial_A(\cdot)], \qquad \partial_a[\partial_b(\cdot)] = \partial_b[\partial_a(\cdot)]. \tag{3}$$

Let $\mathbf{X} \in B_0$ and $\mathbf{x} \in B$ denote position vectors in Euclidean space. Referential and spatial basis vectors are then written, respectively, as

$$\mathbf{G}_A(X) = \partial_A \mathbf{X}, \quad \mathbf{g}_a(x) = \partial_a \mathbf{x}. \tag{4}$$

Reciprocal basis vectors are $\mathbf{G}^A(X)$ and $\mathbf{g}^a(x)$. Scalar products of basis vectors and their reciprocals are

$$\langle \mathbf{G}^A, \mathbf{G}_B \rangle = \delta^A_B, \quad \langle \mathbf{g}^a, \mathbf{g}_b \rangle = \delta^a_b. \tag{5}$$

Kronecker delta symbols are δ^A_B and δ^a_b. Indices in parentheses are symmetric, indices in square brackets are anti-symmetric, and indices between vertical bars are excluded from (anti-)symmetry operations, e.g.

$$A_{(AB)} = \frac{1}{2}(A_{AB} + A_{BA}), \quad A_{[AB]} = \frac{1}{2}(A_{AB} - A_{BA}), \quad A_{[A|C|B]} = \frac{1}{2}(A_{ACB} - A_{BCA}). \tag{6}$$

From (3),

$$\partial_{[A}[\partial_{B]}(\cdot)] = 0, \qquad \partial_{[a}[\partial_{b]}(\cdot)] = 0. \tag{7}$$

Using (5), the scalar product of a vector $\mathbf{V} = V^A \mathbf{G}_A$ and a covector $\boldsymbol{\alpha} = \alpha_B \mathbf{G}^B$ is

$$\langle \mathbf{V}, \boldsymbol{\alpha} \rangle = V^A \alpha_B \langle \mathbf{G}_A, \mathbf{G}^B \rangle = V^A \alpha_B \delta^B_A = V^A \alpha_A. \tag{8}$$

The tensor product or outer product $\otimes$ obeys

$$(\mathbf{G}^A \otimes \mathbf{G}^B)\mathbf{G}_C = \mathbf{G}^A \langle \mathbf{G}^B, \mathbf{G}_C \rangle. \tag{9}$$

Symmetric metric tensors $\mathbf{G}(X)$ and $\mathbf{g}(x)$ are introduced for respective configurations B_0 and B:

$$\mathbf{G}(X) = G_{AB} \mathbf{G}^A \otimes \mathbf{G}^B = (\mathbf{G}_A \cdot \mathbf{G}_B)\mathbf{G}^A \otimes \mathbf{G}^B, \quad G_{AB} = G_{(AB)}; \tag{10}$$

$$\mathbf{g}(x) = g_{ab} \mathbf{g}^a \otimes \mathbf{g}^b = (\mathbf{g}_a \cdot \mathbf{g}_b)\mathbf{g}^a \otimes \mathbf{g}^b, \quad g_{ab} = g_{(ab)}; \tag{11}$$

where the dot product of vectors $\mathbf{V}$ and $\mathbf{W}$ is

$$\mathbf{V} \cdot \mathbf{W} = V^A W^B (\mathbf{G}_A \cdot \mathbf{G}_B) = V^A W_A. \tag{12}$$

As indicated, metric tensors can be used to lower contravariant indices:

$$V_A = V^B G_{AB}, \quad \mathbf{G}_A = G_{AB} \mathbf{G}^B. \tag{13}$$

Inverses of metric tensors are

$$\mathbf{G}^{-1} = G^{AB} \mathbf{G}_A \otimes \mathbf{G}_B = (\mathbf{G}^A \cdot \mathbf{G}^B)\mathbf{G}_A \otimes \mathbf{G}_B, \quad G^{AB} = G^{(AB)}; \tag{14}$$

$$\mathbf{g}^{-1} = g^{ab} \mathbf{g}_a \otimes \mathbf{g}_b = (\mathbf{g}^a \cdot \mathbf{g}^b)\mathbf{g}_a \otimes \mathbf{g}_b, \quad g^{ab} = g^{(ab)}; \tag{15}$$

where the dot product of covectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \alpha_A \beta_B (\mathbf{G}^A \cdot \mathbf{G}^B) = \alpha^A \beta_A. \tag{16}$$

As indicated, inverse metric tensors can be used to raise covariant indices:

$$V^A = V_B G^{AB}, \quad \mathbf{G}^A = G^{AB} \mathbf{G}_B. \tag{17}$$

Furthermore, from the definition of the inverse operation,

$$G^{AC} G_{CB} = \delta^A_B, \quad g^{ac} g_{cb} = \delta^a_b. \tag{18}$$

Determinants of metric tensors and their inverses are written as

$$G = \det \mathbf{G} = \det(G_{AB}), \quad G^{-1} = \det \mathbf{G}^{-1} = \det(G^{AB}) = 1/G; \tag{19}$$

$$g = \det \mathbf{g} = \det(g_{ab}), \quad g^{-1} = \det \mathbf{g}^{-1} = \det(g^{ab}) = 1/g. \tag{20}$$

Permutation tensors are defined as

$$\epsilon^{ABC} = \frac{1}{\sqrt{G}}\, e^{ABC}, \quad \epsilon_{ABC} = \sqrt{G}\, e_{ABC}; \qquad \epsilon^{abc} = \frac{1}{\sqrt{g}}\, e^{abc}, \quad \epsilon_{abc} = \sqrt{g}\, e_{abc}. \tag{21}$$

Permutation symbols are e^{ABC}, e_{ABC}, e^{abc}, and e_{abc}; definitions and identities involving third-order permutation symbols implicitly assume a three dimensional ambient space. In this paper, metric tensors are assumed positive definite over any volume [39]; however, in certain coordinate systems, determinants of metric tensors or their inverses may be zero or undefined at certain points, lines, or surfaces; for example, along null radial coordinates in spherical or cylindrical systems. Shifter tensors, examples of two-point tensors, can be introduced in Euclidean space [30, 39]:

$$g^a_A(x, X) = \langle \mathbf{g}^a, \mathbf{G}_A \rangle, \quad g^A_a(x, X) = \langle \mathbf{g}_a, \mathbf{G}^A \rangle; \tag{22}$$

$$g^{aA}(x, X) = \mathbf{g}^a \cdot \mathbf{G}^A = g^{Aa}, \quad g_{aA}(x, X) = \mathbf{g}_a \cdot \mathbf{G}_A = g_{Aa}; \tag{23}$$

$$g^a_B g^B_b = \delta^a_b, \quad g^A_b g^b_B = \delta^A_B; \tag{24}$$

$$g_{aA} = g_{ab} g^b_A = G_{AB} g^B_a = g_{ab} G_{AB} g^{Bb}; \tag{25}$$

$$\det(g^A_a) = 1/\det(g^a_A) = \sqrt{\det(g_{ab})/\det(G_{AB})} = \sqrt{g/G}. \tag{26}$$

Vector $\mathbf{V}$ and covector $\boldsymbol{\alpha}$ can then be written

$$\mathbf{V} = V^A \mathbf{G}_A = (V^a g^A_a)(g^b_A \mathbf{g}_b) = V^a \delta^b_a \mathbf{g}_b = V^a \mathbf{g}_a; \tag{27}$$

$$\boldsymbol{\alpha} = \alpha_A \mathbf{G}^A = (\alpha_a g^a_A)(g^A_b \mathbf{g}^b) = \alpha_a \delta^a_b \mathbf{g}^b = \alpha_a \mathbf{g}^a. \tag{28}$$

The following rules for shifting of basis vectors are implied:

$$\mathbf{g}_a = g^A_a \mathbf{G}_A, \quad \mathbf{g}^a = g^a_A \mathbf{G}^A; \quad \mathbf{G}_A = g^a_A \mathbf{g}_a, \quad \mathbf{G}^A = g^A_a \mathbf{g}^a. \tag{29}$$

2.2. Linear connections

Discussed in what follows are connection coefficients and covariant differentiation, torsion and curvature tensors, Riemannian geometry, and features of Euclidean space.

2.2.1. Covariant derivatives. Associated with a linear connection in configuration B_0 is the covariant derivative operator ∇. The covariant derivative assigns to two vector fields $\mathbf{V}$, $\mathbf{W}$ a third vector field $\nabla_{\mathbf{V}}\mathbf{W}$, called the covariant derivative of $\mathbf{W}$ along $\mathbf{V}$. In coordinates,

$$\nabla_{\mathbf{V}}\mathbf{W} = (V^B \partial_B W^A + \Gamma^{\cdot\cdot A}_{BC} W^C V^B)\mathbf{G}_A. \tag{30}$$

In n-dimensional space, n^3 entries of the object $\Gamma^{\cdot\cdot A}_{BC}$ are called connection coefficients. The covariant derivative is applied to components of vectors and tensors of higher order as [27]

$$\nabla_N A^{A\ldots F}_{G\ldots M} = \partial_N A^{A\ldots F}_{G\ldots M} + \Gamma^{\cdot\cdot A}_{NR} A^{R\ldots F}_{G\ldots M} + \cdots + \Gamma^{\cdot\cdot F}_{NR} A^{A\ldots R}_{G\ldots M} - \Gamma^{\cdot\cdot R}_{NG} A^{A\ldots F}_{R\ldots M} - \cdots - \Gamma^{\cdot\cdot R}_{NM} A^{A\ldots F}_{G\ldots R}. \tag{31}$$

Indices of covariant differentiation on the left-hand side correspond to those of partial differentiation and the first covariant index of the connection coefficients on the right-hand side. The covariant derivative of a scalar field $A(X)$ is equivalent to its partial derivative, i.e. $\nabla_N A = \partial_N A$. Definitions can be applied analogously to configuration B with coordinates x^a.

2.2.2. Torsion and curvature. The torsion tensor of an arbitrary connection with coefficients $\Gamma_{BC}^{..A}$ is defined in holonomic coordinates as

$$\mathbf{T} = T_{BC}^{..A}\mathbf{G}^B \otimes \mathbf{G}^C \otimes \mathbf{G}_A = \Gamma_{[BC]}^{..A}\mathbf{G}^B \otimes \mathbf{G}^C \otimes \mathbf{G}_A. \tag{32}$$

Components of the torsion tensor are the anti-symmetric covariant components of its corresponding connection: $T_{BC}^{..A} = \Gamma_{[BC]}^{..A}$. Components of the Riemann–Christoffel curvature tensor of this arbitrary connection are

$$R_{BCD}^{...A} = 2\partial_{[B}\Gamma_{C]D}^{..A} + 2\Gamma_{[B|E|}^{..A}\Gamma_{C]D}^{..E}. \tag{33}$$

Both the torsion and curvature transform under a conventional change of basis as true tensors. Different definitions are used for torsion and curvature; those listed here follow [14, 27]. Skew second covariant derivatives of a contravariant vector $\mathbf{V}$ and a covector $\boldsymbol{\alpha}$ can be expressed as [27]

$$\nabla_{[B}\nabla_{C]}V^A = \nabla_{[B}(\partial_{C]}V^A + \Gamma_{C]D}^{..A}V^D) = \frac{1}{2}R_{BCD}^{...A}V^D - T_{BC}^{..D}\nabla_D V^A, \tag{34}$$

$$\nabla_{[B}\nabla_{C]}\alpha_D = \nabla_{[B}(\partial_{C]}\alpha_D - \Gamma_{C]D}^{..A}\alpha_A) = -\frac{1}{2}R_{BCD}^{...A}\alpha_A - T_{BC}^{..A}\nabla_A\alpha_D. \tag{35}$$

2.2.3. Arbitrary holonomic connections. Coefficients of an arbitrary connection in holonomic coordinates can be written [27]

$$\Gamma_{BC}^{..A} = \{_{BC}^{..A}\} + T_{BC}^{..A} - T_{C.B}^{.A.} + T_{.BC}^{A..} + \frac{1}{2}(M_{BC}^{..A} + M_{C.B}^{.A.} - M_{.BC}^{A..}). \tag{36}$$

Here, G_{AB} and its inverse G^{AB} are symmetric, three times differentiable, invertible, but otherwise arbitrary second-order tensors. Christoffel symbols of the tensor G_{AB} are

$$\{_{BC}^{..A}\} = \frac{1}{2}G^{AD}(\partial_B G_{CD} + \partial_C G_{BD} - \partial_D G_{BC}) \qquad \text{(symbols of the second kind);} \tag{37}$$

$$\{_{BC,A}\} = G_{AD}\{_{BC}^{..D}\} \qquad \text{(symbols of the first kind).} \tag{38}$$

The third-order object

$$M_{BC}^{..A} = G^{AD}M_{BCD} = -G^{AD}\nabla_B G_{CD} = G_{CD}\nabla_B G^{AD}, \tag{39}$$

where the final equality follows from $\nabla_B(G_{CD}G^{AD}) = \nabla_B\delta_C^A = 0$. The covariant derivative of G_{AB} follows as

$$\nabla_A G_{BC} = \partial_A G_{BC} - \Gamma_{AB}^{..D}G_{DC} - \Gamma_{AC}^{..D}G_{BD} = -M_{ABC} = -M_{A(BC)}. \tag{40}$$

When $\nabla_A G_{BC} = 0$ (or when $M_{ABC} = 0$), the connection is said to be metric with respect to G_{AB}, i.e. a metric connection. For a metric connection, covariant differentiation via ∇ and lowering (raising) indices by G_{AB} (G^{AB}) commute.

2.2.4. Riemannian geometry. In Riemannian geometry, by definition, the torsion vanishes and the connection is metric:

$$\Gamma_{BC}^{..A} = \{_{BC}^{..A}\} = \frac{1}{2}G^{AD}(\partial_B G_{CD} + \partial_C G_{BD} - \partial_D G_{BC}), \quad T_{BC}^{..A} = 0, \quad M_{ABC} = 0. \tag{41}$$

In Riemannian geometry, the number of independent components of the Riemann–Christoffel curvature tensor is $n^2(n^2-1)/12$; for example, one component for two-dimensional space and six for three-dimensional space.

2.2.5. Euclidean space. Let $G_{AB}(X) = \mathbf{G}_A \cdot \mathbf{G}_B$ be the metric tensor of the space. The Levi-Civita connection coefficients of G_{AB}, written as $\overset{G}{\Gamma}{}_{BC}^{..A}$, are the associated metric and torsion-free connection coefficients of (41):

$$\overset{G}{\Gamma}{}_{BC}^{..A} = \frac{1}{2}G^{AD}(\partial_B G_{CD} + \partial_C G_{BD} - \partial_D G_{BC}) = \overset{G}{\Gamma}{}_{CB}^{..A}. \tag{42}$$

The superposed G is a descriptive label rather than a free index and is exempt from the summation convention. In Euclidean space, the Riemann–Christoffel curvature tensor of the Levi-Civita connection vanishes identically:

$$\overset{G}{R}{}_{BCD}^{\cdot\cdot\cdot A} = 2\left(\partial_{[B}\overset{G}{\Gamma}{}_{C]D}^{\cdot\cdot A} + \overset{G}{\Gamma}{}_{[B|E|}^{\cdot\cdot A}\overset{G}{\Gamma}{}_{C]D}^{\cdot\cdot E}\right) = 0. \tag{43}$$

In n-dimensional Euclidean space, a transformation to a n-dimensional Cartesian coordinate system is admitted a each point X such that $G_{AB}(X) \to \delta_{AB}$, where δ_{AB} are covariant Kronecker delta symbols. When the curvature tensor of the connection vanishes, the space is said to be intrinsically flat; otherwise, the space is said to be intrinsically curved. In two dimensions, a cylindrical shell is intrinsically flat, while a spherical shell is intrinsically curved.

Henceforward, reference configuration B_0 is treated as a three-dimensional Euclidean space. More precisely, in this manuscript, when a configuration is identified with n-dimensional Euclidean space, a simply connected deformable body of finite size in that configuration is assumed to occupy an open region of infinitely extended n-dimensional Euclidean vector space. In the interest of brevity, such a configuration is simply labeled a Euclidean space.

The covariant derivative associated with the Levi-Civita connection on B_0 is written as

$$\overset{G}{\nabla}_A(\cdot) = (\cdot)_{;A}. \tag{44}$$

Covariant derivatives of basis vectors and their reciprocals vanish, leading to

$$\mathbf{G}_{A;B} = \partial_B\mathbf{G}_A - \overset{G}{\Gamma}{}_{BA}^{\cdot\cdot C}\mathbf{G}_C = 0 \Rightarrow \partial_B\mathbf{G}_A = \overset{G}{\Gamma}{}_{BA}^{\cdot\cdot C}\mathbf{G}_C, \tag{45}$$

$$\mathbf{G}^A_{;B} = \partial_B\mathbf{G}^A + \overset{G}{\Gamma}{}_{BC}^{\cdot\cdot A}\mathbf{G}^C = 0 \Rightarrow \partial_B\mathbf{G}^A = -\overset{G}{\Gamma}{}_{BC}^{\cdot\cdot A}\mathbf{G}^C. \tag{46}$$

It follows that Christoffel symbols can be computed as

$$\overset{G}{\Gamma}{}_{BC}^{\cdot\cdot A} = \overset{G}{\Gamma}{}_{BC}^{\cdot\cdot D}\delta^A_D = \overset{G}{\Gamma}{}_{BC}^{\cdot\cdot D}\langle\mathbf{G}_D, \mathbf{G}^A\rangle = \langle\partial_B\mathbf{G}_C, \mathbf{G}^A\rangle. \tag{47}$$

The partial derivative of the metric tensor is

$$\partial_C G_{AB} = \partial_C(\mathbf{G}_A \cdot \mathbf{G}_B) = 2\,\overset{G}{\Gamma}{}_{C(A}^{\cdot\cdot D}G_{B)D}. \tag{48}$$

Because the Levi-Civita connection is a metric connection,

$$G_{AB;C} = \partial_C G_{AB} - \overset{G}{\Gamma}{}_{CA}^{\cdot\cdot D}G_{DB} - \overset{G}{\Gamma}{}_{CB}^{\cdot\cdot D}G_{AD} = 0. \tag{49}$$

From the symmetry of the Levi-Civita connection [or (4) and (7)], skew partial derivatives of basis vectors vanish:

$$\partial_{[B}\mathbf{G}_{A]} = \overset{G}{\Gamma}{}_{[BA]}^{\cdot\cdot C}\mathbf{G}_C = \partial_{[B}\partial_{A]}\mathbf{X} = 0. \tag{50}$$

The following identity will be used for the derivative of the determinant of a non-singular but otherwise arbitrary second-order tensor $\mathbf{A}$ [30]:

$$\frac{\partial \det\mathbf{A}}{\partial A^A_{\cdot B}} = A^{-1B}{}_{\cdot A}\det\mathbf{A}. \tag{51}$$

Applying this identity to the determinant of the metric tensor,

$$\partial_A(\ln\sqrt{G}) = \overset{G}{\Gamma}{}_{BA}^{\cdot\cdot B}. \tag{52}$$

Now consider the spatial configuration B, which is also henceforward identified with Euclidean space, with metric tensor components $g_{ab}(x) = \mathbf{g}_a \cdot \mathbf{g}_b$. The spatial Levi-Civita connection is

$$\overset{g}{\Gamma}{}_{bc}^{\cdot\cdot a} = \frac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) = \overset{g}{\Gamma}{}_{cb}^{\cdot\cdot a}. \tag{53}$$

Superposed g is a descriptive label rather than a free index and is exempt from the summation convention. In Euclidean space, the Riemann–Christoffel curvature tensor vanishes:

$$\overset{g}{R}{}_{bcd}^{\cdot\cdot\cdot a} = 2\left(\partial_{[b}\overset{g}{\Gamma}{}_{c]d}^{\cdot\cdot a} + \overset{g}{\Gamma}{}_{[b|e|}^{\cdot\cdot a}\overset{g}{\Gamma}{}_{c]d}^{\cdot\cdot e}\right) = 0. \tag{54}$$

The covariant derivative associated with the Levi-Civita connection on B is written as

$$\overset{g}{\nabla}_a(\cdot) = (\cdot)_{;a}. \tag{55}$$

Analogously to identities for the reference configuration,

$$\mathbf{g}_{a;b} = 0 \Rightarrow \partial_b\mathbf{g}_a = \overset{g}{\Gamma}{}_{ba}^{\cdot\cdot c}\mathbf{g}_c, \tag{56}$$

$$\mathbf{g}^a_{;b} = 0 \Rightarrow \partial_b\mathbf{g}^a = -\overset{g}{\Gamma}{}_{bc}^{\cdot\cdot a}\mathbf{g}^c, \tag{57}$$

$$\overset{g}{\Gamma}{}_{bc}^{\cdot\cdot a} = \langle\partial_b\mathbf{g}_c, \mathbf{g}^a\rangle, \tag{58}$$

$$\partial_{[b}\mathbf{g}_{a]} = \overset{g}{\Gamma}{}_{[ba]}^{\cdot\cdot c}\mathbf{g}_c = \partial_{[b}\partial_{a]}\mathbf{x} = 0, \tag{59}$$

$$\partial_c g_{ab} = 2\overset{g}{\Gamma}{}_{c(a}^{\cdot\cdot d}g_{b)d}, \tag{60}$$

$$\partial_a(\ln\sqrt{g}) = \overset{g}{\Gamma}{}_{ba}^{\cdot\cdot b}, \tag{61}$$

$$g_{ab;c} = 0. \tag{62}$$

2.3. Differentiable operators

Differential operators, specifically the gradient, divergence, curl, and Laplacian are defined for holonomic coordinates in Euclidean space. Partial and total covariant derivatives of two-point tensors are introduced.

2.3.1. Gradient, divergence, curl, and Laplacian. The description that follows in this subsection is framed in Euclidean reference configuration B_0 spanned by holonomic coordinates X^A. Analogous definitions and identities apply for a description in the spatial configuration. The gradient of a scalar function $f(X)$ is equivalent to its partial derivative:

$$\overset{G}{\nabla} f = \overset{G}{\nabla}_A f\mathbf{G}^A = f_{;A}\mathbf{G}^A = \partial_A f\mathbf{G}^A. \tag{63}$$

The gradient of a vector field $\mathbf{V}(X) = V^A\mathbf{G}_A$ is

$$\overset{G}{\nabla}\mathbf{V} = \partial_B\mathbf{V}\otimes\mathbf{G}^B = \left(\partial_B V^A + \overset{G}{\Gamma}{}_{BC}^{\cdot\cdot A}V^C\right)\mathbf{G}_A\otimes\mathbf{G}^B = V^A_{;B}\mathbf{G}_A\otimes\mathbf{G}^B = \overset{G}{\nabla}_A V^B\mathbf{G}^A\otimes\mathbf{G}^B. \tag{64}$$

The trace of a second-order tensor $\mathbf{A}$ is

$$\mathrm{tr}\mathbf{A} = A^A_{\cdot A} = A^{AB}G_{AB} = A_{AB}G^{AB}. \tag{65}$$

The divergence of a vector field $\mathbf{V}(X) = V^A\mathbf{G}_A$ is

$$\langle\overset{G}{\nabla}, \mathbf{V}\rangle = \mathrm{tr}\,(\overset{G}{\nabla}\mathbf{V}) = \langle\partial_A\mathbf{V}, \mathbf{G}^A\rangle = V^A_{;A} = \partial_A V^A + \overset{G}{\Gamma}{}_{AB}^{\cdot\cdot A}V^B = \frac{1}{\sqrt{G}}\partial_A(\sqrt{G}V^A), \tag{66}$$

where the final equality follows from (52). The vector cross product $\times$ obeys, for two vectors $\mathbf{V}$ and $\mathbf{W}$ and two covectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$,

$$\mathbf{V}\times\mathbf{W} = \epsilon_{ABC}V^BW^C\mathbf{G}^A, \qquad \boldsymbol{\alpha}\times\boldsymbol{\beta} = \epsilon^{ABC}\alpha_B\beta_C\mathbf{G}_A. \tag{67}$$

The curl of a covariant vector field $\boldsymbol{\alpha}(X)$ obeys

$$\overset{G}{\nabla}\times\boldsymbol{\alpha} = \mathbf{G}^A\times\partial_A(\alpha_B\mathbf{G}^B) = \mathbf{G}^A\times\mathbf{G}^B\alpha_{B;A} = \epsilon^{ABC}\alpha_{C;B}\mathbf{G}_A = \epsilon^{ABC}\partial_B\alpha_C\mathbf{G}_A, \tag{68}$$

where the final equality follows from symmetry of the Levi-Civita connection. The Laplacian of a scalar field $f(X)$ is

$$\overset{G}{\nabla}{}^2 f = G^{AB}f_{;AB} = (G^{AB}\partial_A f)_{;B} = \frac{1}{\sqrt{G}}\partial_B(\sqrt{G}G^{AB}\partial_A f). \tag{69}$$

The divergence of the curl of a (co)vector field vanishes identically:

$$\langle \overset{G}{\nabla}, \overset{G}{\nabla} \times \boldsymbol{\alpha}\rangle = \epsilon^{ABC}\alpha_{C\,;BA} = \epsilon^{C[AB]}\alpha_{C\,;(AB)} = 0, \tag{70}$$

as does the curl of the gradient of a scalar field:

$$\overset{G}{\nabla} \times \overset{G}{\nabla} f = \epsilon^{ABC} f_{;CB}\mathbf{G}_A = \epsilon^{A[BC]} f_{;(BC)}\mathbf{G}_A = 0. \tag{71}$$

2.3.2. Partial and total covariant derivatives. Consider a two-point tensor (i.e. double tensor) $\mathbf{A}(X, x)$ of order two:

$$\mathbf{A} = A^a_{.A}\mathbf{g}_a \otimes \mathbf{G}^A. \tag{72}$$

Components of the total covariant derivative of $A^a_{.A}$ are defined as [14, 30, 39]

$$A^a_{.A:B} = (A^a_{.A;B}) + (A^a_{.A;b})\partial_B x^b = \left(\frac{\partial A^a_{.A}}{\partial X^B}\bigg|_x - \overset{G}{\Gamma}{}^{..C}_{BA} A^a_{.C}\right) + \left(\frac{\partial A^a_{.A}}{\partial x^b}\bigg|_X + \overset{g}{\Gamma}{}^{..a}_{bc} A^c_{.A}\right)\frac{\partial x^b}{\partial X^B}. \tag{73}$$

Quantities $\partial_B x^b$ will be identified with components of the deformation gradient in Section 2.4.1. Partial covariant derivatives of $A^a_{.A}$ are defined as usual covariant derivatives with respect to indices in one configuration, with those of the other configuration held fixed:

$$A^a_{.A;B} = \frac{\partial A^a_{.A}}{\partial X^B}\bigg|_x - \overset{G}{\Gamma}{}^{..C}_{BA} A^a_{.C}, \qquad A^a_{.A;b} = \frac{\partial A^a_{.A}}{\partial x^b}\bigg|_X + \overset{g}{\Gamma}{}^{..a}_{bc} A^c_{.A}. \tag{74}$$

From (1), $x^a = x^a(X, t)$. Writing $A^a_{.A}[X, x(X, t)] = A^a_{.A}(X, t)$, the partial derivative of $A^a_{.A}$ at fixed t is

$$\frac{\partial A^a_{.A}}{\partial X^B} = \frac{\partial A^a_{.A}}{\partial X^B}\bigg|_t = \frac{\partial A^a_{.A}}{\partial X^B}\bigg|_x + \frac{\partial A^a_{.A}}{\partial x^b}\bigg|_X \partial_B x^b. \tag{75}$$

With this more conventional notation, (73) becomes

$$A^a_{.A:B} = \frac{\partial A^a_{.A}}{\partial X^B} - \overset{G}{\Gamma}{}^{..C}_{BA} A^a_{.C} + \overset{g}{\Gamma}{}^{..a}_{bc} A^c_{.A}\partial_B x^b. \tag{76}$$

The total covariant derivative can also be obtained directly by inspection of the (material) gradient of $\mathbf{A}$:

$$\overset{G}{\nabla}\mathbf{A} = \partial_B \mathbf{A} \otimes \mathbf{G}^B = \partial_B(A^a_{.A}\mathbf{g}_a \otimes \mathbf{G}^A) \otimes \mathbf{G}^B = A^a_{.A:B}\mathbf{g}_a \otimes \mathbf{G}^A \otimes \mathbf{G}^B. \tag{77}$$

The total covariant derivative can be extended to two-point tensors of arbitrary order as [30]

$$(A^{a\ldots e}_{A\ldots E})_{:K} = (A^{a\ldots e}_{A\ldots E})_{;K} + (A^{a\ldots e}_{A\ldots E})_{;k}\partial_K x^k, \tag{78}$$

where partial covariant derivatives are taken with respect to indices referred a single configuration, as in (74). From the definition of the total covariant derivative, noting that $\partial_K x^k\, \partial_l X^K = \delta^k_l$, total covariant derivatives map between configurations like partial derivatives:

$$(A^{a\ldots e}_{A\ldots E})_{:K}\partial_k X^K = (A^{a\ldots e}_{A\ldots E})_{;K}\partial_k X^K + (A^{a\ldots e}_{A\ldots E})_{;l}\partial_K x^l\, \partial_k X^K = (A^{a\ldots e}_{A\ldots E})_{:k}. \tag{79}$$

2.4. Kinematics of integrable deformation

The deformation gradient and its inverse are introduced. The Jacobian determinant associated with volume changes is defined, and Piola's identities are listed.

2.4.1. Deformation gradient. The deformation gradient $\mathbf{F}$ is the two-point tensor with components, from (1),

$$\mathbf{F} = F^a_{\cdot A}\mathbf{g}_a \otimes \mathbf{G}^A, \tag{80}$$

$$F^a_{\cdot A}(X,t) = \frac{\partial \varphi^a(X,t)}{\partial X^A} = \frac{\partial x^a(X,t)}{\partial X^A} = x^a_{,A} = \partial_A x^a. \tag{81}$$

Similarly, the inverse deformation gradient and its components are

$$\mathbf{F} = F^{-1A}_{\cdot a}\mathbf{G}_A \otimes \mathbf{g}^a, \tag{82}$$

$$F^{-1A}_{\cdot a}(x,t) = \frac{\partial \Phi^A(x,t)}{\partial x^a} = \frac{\partial X^A(x,t)}{\partial x^a} = X^A_{,a} = \partial_a X^A. \tag{83}$$

From differentiability and invertibility properties of the mappings in (1), $\det \mathbf{F} \neq 0$ and $\det \mathbf{F}^{-1} \neq 0$. By definition,

$$F^a_{\cdot A}F^{-1A}_{\cdot b} = \delta^a_b, \qquad F^{-1A}_{\cdot a}F^a_{\cdot B} = \delta^A_B. \tag{84}$$

Partial differentiation (holding time t fixed) proceeds as

$$\partial_A(\cdot) = \partial_a(\cdot)\partial_A x^a = \partial_a(\cdot)F^a_{\cdot A}, \qquad \partial_a(\cdot) = \partial_A(\cdot)\partial_a X^A = \partial_A(\cdot)F^{-1A}_{\cdot a}. \tag{85}$$

Consider a differential line element $d\mathbf{X}$ in the reference configuration. Such an element is mapped to its representation in the current configuration $d\mathbf{x}$ via the Taylor series [14, 41]

$$dx^a(X) = (F^a_{\cdot A})\Big|_X dX^A + \frac{1}{2!}(F^a_{\cdot A:B})\Big|_X dX^A\,dX^B + \frac{1}{3!}(F^a_{\cdot A:BC})\Big|_X dX^A\,dX^B\,dX^C + \cdots, \tag{86}$$

where components of the total covariant derivative of $\mathbf{F}$ are defined as in (76):

$$F^a_{\cdot A:B} = \frac{\partial F^a_{\cdot A}}{\partial X^B} - \overset{G}{\Gamma}{}^{\cdot\cdot C}_{BA}F^a_{\cdot C} + \overset{g}{\Gamma}{}^{\cdot\cdot a}_{bc}F^c_{\cdot A}F^b_{\cdot B} = \partial_B(\partial_A x^a) - \overset{G}{\Gamma}{}^{\cdot\cdot C}_{BA}\partial_C x^a + \overset{g}{\Gamma}{}^{\cdot\cdot a}_{bc}\partial_A x^c \partial_B x^b = x^a_{:AB}. \tag{87}$$

Third-order position gradient follows likewise as

$$F^a_{\cdot A:BC} = (F^a_{\cdot A:B})_{:C} = (x^a_{:AB})_{:C} = x^a_{:ABC} = \partial_C[\partial_B(\partial_A x^a)] + \cdots. \tag{88}$$

From the identity $\partial_A[\partial_B(\cdot)] = \partial_B[\partial_A(\cdot)]$ of (3) and symmetry of the (torsion-free) Levi-Civita connection coefficients in both reference and current configurations, it follows that

$$F^a_{\cdot A:B} = F^a_{\cdot B:A} = F^a_{\cdot(A:B)}, \quad F^a_{\cdot A:BC} = F^a_{\cdot(A:BC)}. \tag{89}$$

To first order in $d\mathbf{X}$, (86) is

$$dx^a(X) = (\partial_A x^a)\Big|_X dX^A = (F^a_{\cdot A})\Big|_X dX^A, \quad d\mathbf{x} = \mathbf{F}\,d\mathbf{X}. \tag{90}$$

Relationship (90) is the usual assumption in classical continuum field theories [37].

2.4.2. Jacobian determinant. Jacobian determinant J relates differential volume elements in reference and current configurations:

$$dv = J\,dV. \tag{91}$$

Differential volume elements in the reference (dV) and current configuration (dv) are written symbolically as

$$dV = \sqrt{G}\,dX^1\,dX^2\,dX^3 \subset B_0, \quad dv = \sqrt{g}\,dx^1\,dx^2\,dx^3 \subset B. \tag{92}$$

The Jacobian determinant $J[\mathbf{F}(X,t), g(x), G(X)]$ is, from (21) [14, 37–39]

$$J = \frac{1}{6}\epsilon^{ABC}\epsilon_{abc}\, F^a_{\cdot A}F^b_{\cdot B}F^c_{\cdot C} = \frac{1}{6}\sqrt{g/G}\; e^{ABC}e_{abc}F^a_{\cdot A}F^b_{\cdot B}F^c_{\cdot C} = \sqrt{g/G}\;\det\mathbf{F} = \sqrt{\frac{\det(g_{ab})}{\det(G_{AB})}}\;\det(\partial_A x^a). \tag{93}$$

From similar arguments, inverse Jacobian determinant $J^{-1}[\mathbf{F}^{-1}(x,t), G(X), g(x)] = 1/J$ is

$$J^{-1} = \frac{1}{6}\epsilon^{abc}\epsilon_{ABC}\; F^{-1A}_{\quad\cdot a}F^{-1B}_{\quad\cdot b}F^{-1C}_{\quad\cdot c} = \sqrt{\det(G_{AB})/\det(g_{ab})}\;\det(\partial_a X^A). \tag{94}$$

When motion is restricted to rigid translation (or no motion at all), then $F^a_{\cdot A} = g^a_A$ [the shifter of (22)], $\mathbf{F} = g^a_A\mathbf{g}_a \otimes \mathbf{G}^A$, and $J = \sqrt{g/G}\det(g^a_A) = 1$ follows from (26). Since volume remains positive, $J > 0$ and $\det\mathbf{F} > 0$. From (51),

$$\frac{\partial J}{\partial F^a_{\cdot A}} = JF^{-1A}_{\quad\cdot a}, \qquad \frac{\partial J^{-1}}{\partial F^{-1A}_{\quad\cdot a}} = J^{-1}F^a_{\cdot A}. \tag{95}$$

One of Piola's identities is derived by taking the divergence of the second expression of (95) [14]:

$$(J^{-1}F^a_{\cdot A})_{:a} = \partial_a(J^{-1}F^a_{\cdot A}) + J^{-1}\overset{g}{\Gamma}{}^{\cdot\cdot a}_{ab}F^b_{\cdot A} - J^{-1}\overset{G}{\Gamma}{}^{\cdot\cdot B}_{BA} = J^{-1}F^{-1B}_{\quad\cdot a}\,[\partial_B(\partial_A x^a) - \partial_A(\partial_B x^a)] = 0. \tag{96}$$

The same steps performed on (95) with reference and spatial coordinates interchanged produce

$$(JF^{-1A}_{\quad\cdot a})_{:A} = JF^b_{\cdot A}\left[\partial_b F^{-1A}_{\quad\cdot a} - \partial_a F^{-1A}_{\quad\cdot b}\right] = JF^b_{\cdot A}\left[\partial_b(\partial_a X^A) - \partial_a(\partial_b X^A)\right] = 0. \tag{97}$$

Let the vector field $\mathbf{A}(X) = A^A\mathbf{G}_A$ be the Piola transform of $\mathbf{a}(x) = a^a\mathbf{g}_a$:

$$A^A = JF^{-1A}_{\quad\cdot a}a^a. \tag{98}$$

Taking the divergence of (98) and applying the product rule for covariant differentiation along with (97) gives

$$A^A_{;A} = A^A_{:A} = (JF^{-1A}_{\quad\cdot a})_{:A}a^a + JF^{-1A}_{\quad\cdot a}a^a_{:A} = JF^{-1A}_{\quad\cdot a}a^a_{:A} = Ja^a_{:a} = Ja^a_{;a}. \tag{99}$$

3. Anholonomic deformation

3.1. Anholonomic spaces and geometric interpretation

The deformation gradient is split multiplicatively into two terms, neither of which is necessarily integrable. Geometric consequences of such a construction are considered, including rules for differentiation, coordinate systems, metric tensors, and connection coefficients convected from reference or current configurations to anholonomic space.

3.1.1. Two-term decomposition of deformation gradient. Consider a multiplicative split of the (total) deformation gradient $\mathbf{F}$ into two terms:

$$\mathbf{F} = \bar{\mathbf{F}}\tilde{\mathbf{F}}, \tag{100}$$

or in indicial notation, letting the Greek index $\alpha = 1, \ldots n$, where n is the dimension of Euclidean space,

$$\frac{\partial\varphi^a}{\partial X^A} = \partial_A x^a = F^a_{\cdot A} = \bar{F}^a_{\cdot\alpha}\tilde{F}^\alpha_{\cdot A}. \tag{101}$$

In coordinates, terms on the right-hand side of (100) can be written

$$\bar{\mathbf{F}}(X,t) = \bar{F}^a_{\cdot\alpha}\mathbf{g}_a \otimes \tilde{\mathbf{g}}^\alpha, \quad \tilde{\mathbf{F}}(X,t) = \tilde{F}^\alpha_{\cdot A}\tilde{\mathbf{g}}_\alpha \otimes \mathbf{G}^A. \tag{102}$$

Basis vectors $\tilde{\mathbf{g}}_\alpha$ and their reciprocals $\tilde{\mathbf{g}}^\alpha$ will be described in detail later. Both $\bar{\mathbf{F}}$ and $\tilde{\mathbf{F}}$ are second-order, two-point tensor fields with positive determinants:

$$\det\bar{\mathbf{F}} = \frac{1}{6}e_{abc}e^{\alpha\beta\chi}\bar{F}^a_{\cdot\alpha}\bar{F}^b_{\cdot\beta}\bar{F}^c_{\cdot\chi} > 0, \qquad \det\tilde{\mathbf{F}} = \frac{1}{6}e_{\alpha\beta\chi}e^{ABC}\tilde{F}^\alpha_{\cdot A}\tilde{F}^\beta_{\cdot B}\tilde{F}^\chi_{\cdot C} > 0. \tag{103}$$

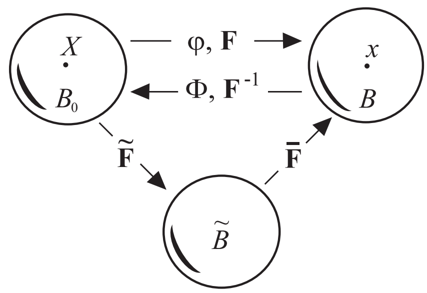

Figure 1. Mappings among reference, intermediate, and current configurations of a deformable body.

Permutation symbols are $e^{\alpha\beta\chi}$ and $e_{\alpha\beta\chi}$; relations involving these implicitly assume $n = 3$. Inverting (100)–(102),

$$\mathbf{F}^{-1} = \tilde{\mathbf{F}}^{-1}\bar{\mathbf{F}}^{-1}, \qquad \partial_a X^A = F^{-1A}_{\bullet a} = \tilde{F}^{-1A}_{\bullet\alpha}\bar{F}^{-1\alpha}_{\bullet a}; \tag{104}$$

$$\tilde{\mathbf{F}}^{-1}(x,t) = \tilde{F}^{-1A}_{\bullet\alpha}\mathbf{G}_A \otimes \tilde{\mathbf{g}}^{\alpha}, \qquad \bar{\mathbf{F}}^{-1}(x,t) = \bar{F}^{-1\alpha}_{\bullet a}\tilde{\mathbf{g}}_{\alpha} \otimes \mathbf{g}^{a}. \tag{105}$$

Furthermore, from the definition of the inverse,

$$\tilde{F}^{\alpha}_{\bullet B}\tilde{F}^{-1A}_{\bullet\alpha} = \delta^A_B, \qquad \bar{F}^{a}_{\bullet\alpha}\bar{F}^{-1\alpha}_{\bullet b} = \delta^a_b, \qquad \tilde{F}^{\alpha}_{\bullet A}\tilde{F}^{-1A}_{\bullet\beta} = \bar{F}^{a}_{\bullet\beta}\bar{F}^{-1\alpha}_{\bullet a} = \delta^{\alpha}_{\beta}, \tag{106}$$

with Kronecker delta symbols δ^{α}_{β}. The target space of $\tilde{\mathbf{F}}$ and $\bar{\mathbf{F}}^{-1}$ is intermediate configuration $\tilde{B}$. Figure 1 illustrates deformation mappings between reference, intermediate, and current configurations entering (100).

A multiplicative decomposition such as (100) can be used to represent various physical behaviors. Usually, $\bar{\mathbf{F}}$ is associated with locally recoverable elastic deformation, such that each material element in configuration $\tilde{B}$ is locally unloaded or stress-free. In geometrically nonlinear crystal mechanics [2, 3, 9, 11, 14, 25, 33, 42, 43], $\bar{\mathbf{F}}$ is associated with lattice (i.e. thermoelastic) deformation, and $\tilde{\mathbf{F}}$ is associated with plastic (i.e. dislocation slip enabled) deformation. In multiplicative descriptions involving explicit thermal deformation [14, 15, 25], the total compatible thermoelastic deformation $\mathbf{F}$ is split into locally recoverable elastic deformation $\bar{\mathbf{F}}$ and stress-free thermal deformation $\tilde{\mathbf{F}}$. In theories of twinning [19, 20], $\tilde{\mathbf{F}}$ represents stress-free twinning shear. In theories of porous or damaged media [18], $\tilde{\mathbf{F}}$ represents volumetric expansion associated with voids. In theories of growth in biomaterials [16, 25], $\tilde{\mathbf{F}}$ represents stretching and mass changes associated with tissue structure evolution.

A decomposition somewhat alternative to (100) enables description of materials containing initial residual stresses and various kinds of defects. In the continuum theory of inhomogeneous bodies of Noll [6], a decomposition $\hat{\mathbf{F}} = (\nabla\gamma)\mathbf{K}^{-1}$ is proposed, where $\mathbf{K}$ may denote a generally anholonomic mapping from a residually stressed intermediate configuration to an unstressed, but generally disconnected, reference configuration. The elastic deformation associated with stresses superimposed on the intermediate configuration is represented by the integrable (i.e. compatible) deformation gradient $\nabla\gamma$, where γ is the motion associated with this elastic deformation gradient. Mapping $\hat{\mathbf{F}}$, like $\mathbf{K}$, need not be integrable. Wang and Truesdell [23] applied similar ideas towards finding universal solutions for certain problems involving pre-stressed laminated elastic bodies.

3.1.2. Anholonomicity conditions and partial differentiation. It is assumed that $\tilde{\mathbf{F}}(X,t)$ is at least once continuously differentiable with respect to referential coordinates X^A. Single-valued coordinates $\tilde{x}^{\alpha}(X,t)$ referred to intermediate configuration $\tilde{B}$ that are at least two times differentiable with respect to X^A exist if and only if the following integrability conditions apply [27]:

$$\partial_A\tilde{F}^{\alpha}_{\bullet B} = \partial_B\tilde{F}^{\alpha}_{\bullet A} \Leftrightarrow \frac{\partial^2\tilde{x}^{\alpha}}{\partial X^A\partial X^B} = \frac{\partial^2\tilde{x}^{\alpha}}{\partial X^B\partial X^A}. \tag{107}$$

When (107) applies globally (i.e. for all $X \in B_0$, which presumably is associated with a simply connected body) intermediate configuration $\tilde{B}$ is said to be holonomic, and then like the reference configuration B_0, is labeled a Euclidean n-dimensional space:

$$\partial_{[A}\tilde{F}^{\alpha}_{\bullet B]} = 0 \Leftrightarrow \tilde{F}^{\alpha}_{\bullet A} = \partial_A\tilde{x}^{\alpha}. \tag{108}$$

Otherwise, single-valued coordinates $\tilde{x}^{\alpha}(X,t)$ continuously (two times) differentiable with respect to X^A do not exist in intermediate configuration $\tilde{B}$, which is then termed an anholonomic space:

$$\partial_{[A}\tilde{F}^{\alpha}_{\centerdot B]} \neq 0 \Leftrightarrow \tilde{x}^{\alpha}(X,t) \text{ anholonomic}. \tag{109}$$

The domain of $\tilde{\mathbf{F}}$ must be simply connected to ensure that the left equality of (108) constitutes sufficient conditions for existence of a uniform covering $\tilde{x}^{\alpha}$ of that domain, in which case both $\tilde{x}^{\alpha}$ and X^A are single valued for all $X \in B_0$. When (108) (respectively, (109)) applies only over local simply connected regions of B_0, then deformation map $\tilde{\mathbf{F}}$ is designated as holonomic (respectively, anholonomic) only in those regions. Regardless of which of (108) or (109) applies, partial differentiation with respect to intermediate coordinates is defined as follows [5, 27, 28, 31]:

$$\partial_{\alpha}(\cdot) = \partial_A(\cdot)\tilde{F}^{-1A}_{\centerdot\alpha}. \tag{110}$$

Arguments in (107)–(110) can be repeated for $\bar{\mathbf{F}}^{-1}(x,t)$, which is assumed continuously differentiable with respect to spatial coordinates x^a. Single-valued coordinates $\tilde{x}^{\alpha}(x,t)$ referred to intermediate configuration $\tilde{B}$ that are continuous and at least two times differentiable with respect to x^a exist if and only if

$$\partial_a\bar{F}^{-1\alpha}_{\centerdot b} = \partial_b\bar{F}^{-1\alpha}_{\centerdot a} \Leftrightarrow \frac{\partial^2\tilde{x}^{\alpha}}{\partial x^a\partial x^b} = \frac{\partial^2\tilde{x}^{\alpha}}{\partial x^b\partial x^a}. \tag{111}$$

When (111) applies globally on B, intermediate configuration $\tilde{B}$ is holonomic:

$$\partial_{[a}\bar{F}^{-1\alpha}_{\centerdot b]} = 0 \Leftrightarrow \bar{F}^{-1\alpha}_{\centerdot a} = \partial_a\tilde{x}^{\alpha}. \tag{112}$$

Otherwise,

$$\partial_{[a}\bar{F}^{-1\alpha}_{\centerdot b]} \neq 0 \Leftrightarrow \tilde{x}^{\alpha}(x,t) \text{ anholonomic}. \tag{113}$$

Partial differentiation with respect to intermediate coordinates also obeys

$$\partial_{\alpha}(\cdot) = \partial_a(\cdot)\bar{F}^{a}_{\centerdot\alpha}. \tag{114}$$

Verification that (110) and (114) are equivalent is straightforward using (85) and (101):

$$\partial_{\alpha}(\cdot) = \partial_A(\cdot)\tilde{F}^{-1A}_{\centerdot\alpha} = \partial_a(\cdot)\,\partial_A x^a\,\tilde{F}^{-1A}_{\centerdot\alpha} = \partial_a(\cdot)F^{a}_{\centerdot A}\tilde{F}^{-1A}_{\centerdot\alpha} = \partial_a(\cdot)\bar{F}^{a}_{\centerdot\alpha}. \tag{115}$$

Second partial anholonomic derivatives obey the relations

$$\partial_{\alpha}[\partial_{\beta}(\cdot)] = \partial_A[\partial_B(\cdot)\tilde{F}^{-1B}_{\centerdot\beta}]\tilde{F}^{-1A}_{\centerdot\alpha} = \partial_{\beta}[\partial_{\alpha}(\cdot)] + 2\partial_A(\cdot)\partial_{[\alpha}\tilde{F}^{-1A}_{\centerdot\beta]}. \tag{116}$$

Following similar arguments,

$$\partial_{\alpha}[\partial_{\beta}(\cdot)] = \partial_{\beta}[\partial_{\alpha}(\cdot)] + 2\partial_a(\cdot)\partial_{[\alpha}\bar{F}^{a}_{\centerdot\beta]}. \tag{117}$$

In general, second partial anholonomic differentiation is not symmetric. Explicitly,

$$\partial_{[\alpha}[\partial_{\beta]}(\cdot)] = \frac{1}{2}\{\partial_{\alpha}[\partial_{\beta}(\cdot)] - \partial_{\beta}[\partial_{\alpha}(\cdot)]\} = \partial_A(\cdot)\partial_{[\alpha}\tilde{F}^{-1A}_{\centerdot\beta]} = \partial_a(\cdot)\partial_{[\alpha}\bar{F}^{a}_{\centerdot\beta]}. \tag{118}$$

Only when (108) and (112) apply such that $\tilde{B}$ is holonomic is second partial anholonomic differentiation is symmetric:

$$\tilde{F}^{-1A}_{\centerdot\beta} = \partial_{\beta}X^A \Rightarrow \partial_{[\alpha}\tilde{F}^{-1A}_{\centerdot\beta]} = \partial_{[\alpha}\partial_{\beta]}X^A = 0 \Rightarrow \partial_{\alpha}[\partial_{\beta}(\cdot)] = \partial_{\beta}[\partial_{\alpha}(\cdot)], \tag{119}$$

$$\bar{F}^{a}_{\centerdot\beta} = \partial_{\beta}x^a \Rightarrow \partial_{[\alpha}\bar{F}^{a}_{\centerdot\beta]} = \partial_{[\alpha}\partial_{\beta]}x^a = 0 \Rightarrow \partial_{\alpha}[\partial_{\beta}(\cdot)] = \partial_{\beta}[\partial_{\alpha}(\cdot)]. \tag{120}$$

Even when (108) and (112) apply globally on B_0 and B, respectively, inverse coordinate functions $X^A(\tilde{x},t)$ and $x^a(\tilde{x},t)$ may only be available locally [6].

3.1.3. Anholonomic basis vectors and metric tensors. By definition, basis vectors and their reciprocals in intermediate configuration $\tilde{B}$ obey

$$\langle \tilde{\mathbf{g}}^{\alpha}, \tilde{\mathbf{g}}_{\beta} \rangle = \delta^{\alpha}_{\beta}. \tag{121}$$

A symmetric metric tensor $\tilde{\mathbf{g}}$ on $\tilde{B}$ is defined in components in terms of a scalar or dot product as

$$\tilde{g}_{\alpha\beta} = \tilde{\mathbf{g}}_{\alpha} \cdot \tilde{\mathbf{g}}_{\beta} = \tilde{\mathbf{g}}_{\beta} \cdot \tilde{\mathbf{g}}_{\alpha} = \tilde{g}_{\beta\alpha} = \tilde{g}_{(\alpha\beta)}. \tag{122}$$

The dot product of two generic contravariant vectors $\mathbf{V} = V^{\alpha}\tilde{\mathbf{g}}_{\alpha}$ and $\mathbf{W} = W^{\alpha}\tilde{\mathbf{g}}_{\alpha}$ is computed as

$$\mathbf{V} \cdot \mathbf{W} = V^{\alpha}\tilde{\mathbf{g}}_{\alpha} \cdot W^{\beta}\tilde{\mathbf{g}}_{\beta} = V^{\alpha}W^{\beta}(\tilde{\mathbf{g}}_{\alpha} \cdot \tilde{\mathbf{g}}_{\beta}) = V^{\alpha}\tilde{g}_{\alpha\beta}W^{\beta} = V^{\alpha}W_{\alpha} = V_{\alpha}W^{\alpha}. \tag{123}$$

As indicated, the metric tensor can be used to lower contravariant indices in the usual manner:

$$V_{\alpha} = V^{\beta}\tilde{g}_{\alpha\beta}, \quad \tilde{\mathbf{g}}_{\alpha} = \tilde{g}_{\alpha\beta}\tilde{\mathbf{g}}^{\beta}. \tag{124}$$

Metric $\tilde{\mathbf{g}}$ is assumed to be positive definite, with positive determinant $\tilde{g}$ over any volume (i.e. excluding possible points, lines, or surfaces where $\tilde{g}$ may be zero or undefined):

$$\tilde{g} = \det \tilde{\mathbf{g}} = \det(\tilde{g}_{\alpha\beta}) = \frac{1}{6}e^{\alpha\beta\chi}e^{\delta\epsilon\phi}\tilde{g}_{\alpha\delta}\,\tilde{g}_{\beta\epsilon}\,\tilde{g}_{\chi\phi} > 0. \tag{125}$$

The inverse $\tilde{\mathbf{g}}^{-1}$ with components $\tilde{g}^{\alpha\beta}$ on $\tilde{B}$ obeys, by definition,

$$\tilde{g}^{\alpha\beta} = \tilde{\mathbf{g}}^{\alpha} \cdot \tilde{\mathbf{g}}^{\beta} = \tilde{\mathbf{g}}^{\beta} \cdot \tilde{\mathbf{g}}^{\alpha} = \tilde{g}^{\beta\alpha} = \tilde{g}^{(\alpha\beta)}, \qquad \tilde{g}^{\alpha\chi}\tilde{g}_{\chi\beta} = \delta^{\alpha}_{\beta}. \tag{126}$$

The inverse metric (126) enables the dot product of generic covariant vectors $\boldsymbol{\alpha} = \alpha_{\alpha}\tilde{\mathbf{g}}^{\alpha}$ and $\boldsymbol{\beta} = \beta_{\alpha}\tilde{\mathbf{g}}^{\alpha}$ on $\tilde{B}$:

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \alpha_{\alpha}\tilde{\mathbf{g}}^{\alpha} \cdot \beta_{\beta}\tilde{\mathbf{g}}^{\beta} = \alpha_{\alpha}\beta_{\beta}(\tilde{\mathbf{g}}^{\alpha} \cdot \tilde{\mathbf{g}}^{\beta}) = \alpha_{\alpha}\tilde{g}^{\alpha\beta}\beta_{\beta} = \alpha^{\alpha}\beta_{\alpha} = \alpha_{\alpha}\beta^{\alpha}. \tag{127}$$

Components of $\tilde{\mathbf{g}}^{-1}$ can be used to raise indices as shown above. Permutation tensors in configuration $\tilde{B}$ are defined as

$$\epsilon^{\alpha\beta\chi} = \frac{1}{\sqrt{\tilde{g}}}\,e^{\alpha\beta\chi}, \quad \epsilon_{\alpha\beta\chi} = \sqrt{\tilde{g}}\,e_{\alpha\beta\chi}. \tag{128}$$

Shifter tensors can also be introduced among basis vectors in intermediate and reference or spatial configurations:

$$g^{\alpha}_{A} = \langle \tilde{\mathbf{g}}^{\alpha}, \mathbf{G}_{A} \rangle, \quad g^{A}_{\alpha} = \langle \tilde{\mathbf{g}}_{\alpha}, \mathbf{G}^{A} \rangle; \quad g^{\alpha}_{a} = \langle \tilde{\mathbf{g}}^{\alpha}, \mathbf{g}_{a} \rangle, \quad g^{a}_{\alpha} = \langle \tilde{\mathbf{g}}_{\alpha}, \mathbf{g}^{a} \rangle; \tag{129}$$

$$g^{\alpha A} = \tilde{\mathbf{g}}^{\alpha} \cdot \mathbf{G}^{A} = g^{A\alpha}, \quad g_{\alpha A} = \tilde{\mathbf{g}}_{\alpha} \cdot \mathbf{G}_{A} = g_{A\alpha}; \tag{130}$$

$$g^{\alpha a} = \tilde{\mathbf{g}}^{\alpha} \cdot \mathbf{g}^{a} = g^{a\alpha}, \quad g_{\alpha a} = \tilde{\mathbf{g}}_{\alpha} \cdot \mathbf{g}_{a} = g_{a\alpha}; \tag{131}$$

$$g^{A}_{\alpha}g^{\alpha}_{B} = \delta^{A}_{B}, \quad g^{a}_{\alpha}g^{\alpha}_{b} = \delta^{a}_{b}, \quad g^{\alpha}_{A}g^{A}_{\beta} = g^{\alpha}_{a}g^{a}_{\beta} = \delta^{\alpha}_{\beta}; \tag{132}$$

$$\det(g^{A}_{\alpha}) = 1/\det(g^{\alpha}_{A}) = \sqrt{\det(\tilde{g}_{\alpha\beta})/\det(G_{AB})} = \sqrt{\tilde{g}/G}; \tag{133}$$

$$\det(g^{\alpha}_{a}) = 1/\det(g^{a}_{\alpha}) = \sqrt{\det(g_{ab})/\det(\tilde{g}_{\alpha\beta})} = \sqrt{g/\tilde{g}}. \tag{134}$$

The following rules apply for shifting of basis vectors:

$$\tilde{\mathbf{g}}_{\alpha} = g^{A}_{\alpha}\mathbf{G}_{A} = g^{a}_{\alpha}\mathbf{g}_{a}, \quad \tilde{\mathbf{g}}^{\alpha} = g^{\alpha}_{A}\mathbf{G}^{A} = g^{\alpha}_{a}\mathbf{g}^{a}; \tag{135}$$

$$\mathbf{G}_{A} = g^{\alpha}_{A}\tilde{\mathbf{g}}_{\alpha}, \quad \mathbf{G}^{A} = g^{A}_{\alpha}\tilde{\mathbf{g}}^{\alpha}; \quad \mathbf{g}_{a} = g^{\alpha}_{a}\tilde{\mathbf{g}}_{\alpha}, \quad \mathbf{g}^{a} = g^{a}_{\alpha}\tilde{\mathbf{g}}^{\alpha}. \tag{136}$$

Furthermore, since

$$\mathbf{g}_{a} = g^{\alpha}_{a}\tilde{\mathbf{g}}_{\alpha} = g^{\alpha}_{a}g^{A}_{\alpha}\mathbf{G}_{A} = g^{A}_{a}\mathbf{G}_{A} \Leftrightarrow g^{A}_{a} = g^{\alpha}_{a}g^{A}_{\alpha}, \tag{137}$$

it follows from the product rule of determinants that $\det(g_a^A) = [\det(g_a^\alpha)][\det(g_\alpha^A)] = [\sqrt{g/\tilde{g}}][\sqrt{\tilde{g}/G}] = \sqrt{g/G}$ in agreement with (26).

First consider the case when (108) applies, such that $\tilde{B}$ can be regarded as a Euclidean space and a position vector $\tilde{\mathbf{x}}$ can be assigned to any point $\tilde{x}(X,t) \in \tilde{B}$. In that case, basis vectors can be defined in the usual manner, similarly to (4):

$$\tilde{\mathbf{g}}_\alpha(\tilde{x}) = \partial_\alpha \tilde{\mathbf{x}} = \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{x}^\alpha}. \tag{138}$$

Metric tensor components corresponding to (138) are

$$\tilde{g}_{\alpha\beta}(\tilde{x}) = \partial_\alpha \tilde{\mathbf{x}} \cdot \partial_\beta \tilde{\mathbf{x}}. \tag{139}$$

Any time-independent Euclidean coordinate system (e.g. fixed curvilinear or Cartesian coordinates) can be used for $\tilde{x}^\alpha$ in this case.

Next consider the case when (109) applies, such that single-valued coordinates $\tilde{x}^\alpha(X,t)$ continuously differentiable with respect to X^A do not exist in $\tilde{B}$. In that case, a one-to-one correspondence between a material particle X and an intermediate point $\tilde{x}$ is not applicable. Intermediate basis vectors associated with a given material particle should then be assigned to X rather than $\tilde{x}$, i.e. $\tilde{\mathbf{g}}_\alpha(X)$. Any time-independent coordinate system could be used for $\tilde{\mathbf{g}}_\alpha(X)$ in this case (time-dependent convected systems are considered later). However, it may prove practical to select identical, possibly curvilinear, coordinate systems in configurations B_0 and $\tilde{B}$ [14, 36]:

$$\tilde{\mathbf{g}}_\alpha(X) = \delta_\alpha^A \mathbf{G}_A = \delta_\alpha^A \partial_A \mathbf{X} = \delta_\alpha^A \tilde{F}^\beta_{.A} \partial_\beta \mathbf{X}. \tag{140}$$

Metric tensor components corresponding to (140) are

$$\tilde{g}_{\alpha\beta}(X) = \tilde{\mathbf{g}}_\alpha \cdot \tilde{\mathbf{g}}_\beta = \delta_\alpha^A \mathbf{G}_A \cdot \delta_\beta^B \mathbf{G}_B = \delta_\alpha^A \delta_\beta^B G_{AB}, \tag{141}$$

where $G_{AB}(X)$ is the referential metric tensor of (10). Mixed Kronecker delta symbols are equivalent to shifter tensors for coincident coordinate systems and have the usual meaning, i.e.

$$\delta_\alpha^A = 1 \forall A = \alpha, \quad \delta_\alpha^A = 0 \forall A \neq \alpha. \tag{142}$$

The determinant of the metric is simply

$$\tilde{g}(X) = \det(\tilde{g}_{\alpha\beta}) = \det(\delta_\alpha^A)\det(\delta_\beta^B)\det(G_{AB}) = \det(G_{AB}) = G. \tag{143}$$

Explicitly, shifter tensor components are

$$g_A^\alpha = \langle \tilde{\mathbf{g}}^\alpha, \mathbf{G}_A \rangle = \langle \delta_B^\alpha \mathbf{G}^B, \mathbf{G}_A \rangle = \delta_B^\alpha \delta_A^B = \delta_A^\alpha, \tag{144}$$

with determinant (133) reducing to

$$\det(g_\alpha^A) = 1/\det(g_A^\alpha) = \sqrt{\tilde{g}/G} = 1. \tag{145}$$

Now consider the case when (113) applies, such that single-valued coordinates $\tilde{x}^\alpha(x,t)$ continuously differentiable with respect to x^a do not exist in $\tilde{B}$. In that case, a one-to-one correspondence between a spatial point x and an intermediate point $\tilde{x}$ is not applicable. Intermediate basis vectors associated with a given spatial point should then be assigned to x rather than $\tilde{x}$, i.e. $\tilde{\mathbf{g}}_\alpha(x)$. Any time-independent coordinate system could be used for $\tilde{\mathbf{g}}_\alpha(x)$ in this case. However, a pragmatic choice corresponds to identical coordinate systems in configurations B and $\tilde{B}$ [14]:

$$\tilde{\mathbf{g}}_\alpha(x) = \delta_\alpha^a \mathbf{g}_a = \delta_\alpha^a \partial_a \mathbf{x} = \delta_\alpha^a \bar{F}^{-1\beta}_{.a} \partial_\beta \mathbf{x}. \tag{146}$$

Choices (140) and (146) differ but are not contradictory; in particular, basis vectors in (140) and (146) are related by $\delta_\alpha^a \mathbf{g}_a[x(X,t)] = \delta_\alpha^a g_a^A(x,X)\delta_A^\beta \delta_\beta^B \mathbf{G}_B(X)$. Metric tensor components corresponding to (146) are

$$\tilde{g}_{\alpha\beta}(x) = \tilde{\mathbf{g}}_\alpha \cdot \tilde{\mathbf{g}}_\beta = \delta_\alpha^a \mathbf{g}_a \cdot \delta_\beta^b \mathbf{g}_b = \delta_\alpha^a \delta_\beta^b g_{ab}, \tag{147}$$

where $g_{ab}(x)$ is the spatial metric tensor of (11). Mixed Kronecker delta symbols satisfy

$$\delta^a_\alpha = 1 \forall a = \alpha, \quad \delta^a_\alpha = 0 \forall a \neq \alpha. \tag{148}$$

The determinant of the metric is

$$\tilde{g}(x) = \det(\tilde{g}_{\alpha\beta}) = \det(\delta^a_\alpha)\det(\delta^b_\beta)\det(g_{ab}) = \det(g_{ab}) = g. \tag{149}$$

Explicitly, shifter tensor components are

$$g^a_\alpha = \langle \tilde{\mathbf{g}}_\alpha, \mathbf{g}^a \rangle = \langle \delta^b_\alpha \mathbf{g}_b, \mathbf{g}^a \rangle = \delta^b_\alpha \delta^a_b = \delta^a_\alpha, \tag{150}$$

with determinant of (134) reducing to

$$\det(g^\alpha_a) = 1/\det(g^a_\alpha) = \sqrt{g/\tilde{g}} = 1. \tag{151}$$

Finally, the simplest choice of coordinate system for configuration $\tilde{B}$ is a Cartesian system with constant basis vectors $\mathbf{e}_\alpha$:

$$\tilde{\mathbf{g}}_\alpha = \mathbf{e}_\alpha, \quad \tilde{g}_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \delta_{\alpha\beta}, \quad \tilde{g} = \det(\delta_{\alpha\beta}) = 1. \tag{152}$$

When anholonomicity conditions (109) or (113) apply, such a Cartesian frame is assigned externally to configuration $\tilde{B}$, since in that case $\tilde{B}$ is not a Euclidean space.

3.1.4. Convected anholonomic connection coefficients. Arbitrary connection coefficients $\Gamma^{\cdot\cdot A}_{BC}$ in reference configuration B_0 can be mapped to coefficients $\hat{\Gamma}^{\cdot\cdot\alpha}_{\beta\chi}$ in configuration $\tilde{B}$ via [14, 27]

$$\hat{\Gamma}^{\cdot\cdot\alpha}_{\beta\chi} = \tilde{F}^\alpha_{\cdot A}\tilde{F}^{-1B}_{\cdot\beta}\tilde{F}^{-1C}_{\cdot\chi}\Gamma^{\cdot\cdot A}_{BC} + \tilde{F}^\alpha_{\cdot A}\partial_\beta \tilde{F}^{-1A}_{\cdot\chi} = \tilde{F}^\alpha_{\cdot A}\tilde{F}^{-1B}_{\cdot\beta}\tilde{F}^{-1C}_{\cdot\chi}\Gamma^{\cdot\cdot A}_{BC} - \tilde{F}^{-1B}_{\cdot\beta}\tilde{F}^{-1A}_{\cdot\chi}\partial_B \tilde{F}^\alpha_{\cdot A}, \tag{153}$$

where (110) applies. Torsion tensor components $T^{\cdot\cdot A}_{BC}$ of (32) in configuration B_0 map correspondingly as

$$\hat{T}^{\cdot\cdot\alpha}_{\beta\chi} = \tilde{F}^\alpha_{\cdot A}\tilde{F}^{-1B}_{\cdot\beta}\tilde{F}^{-1C}_{\cdot\chi}T^{\cdot\cdot A}_{BC} = \tilde{F}^\alpha_{\cdot A}\tilde{F}^{-1B}_{\cdot\beta}\tilde{F}^{-1C}_{\cdot\chi}\Gamma^{\cdot\cdot A}_{[BC]} = \hat{\Gamma}^{\cdot\cdot\alpha}_{[\beta\chi]} + \hat{\kappa}^{\cdot\cdot\alpha}_{\beta\chi} = \hat{T}^{\cdot\cdot\alpha}_{[\beta\chi]}, \tag{154}$$

where components of the anholonomic object $\hat{\boldsymbol{\kappa}}$ are defined as [27]

$$\hat{\kappa}^{\cdot\cdot\alpha}_{\beta\chi} = \tilde{F}^{-1A}_{\cdot\beta}\tilde{F}^{-1B}_{\cdot\chi}\partial_{[A}\tilde{F}^\alpha_{\cdot B]} = \hat{\kappa}^{\cdot\cdot\alpha}_{[\beta\chi]}. \tag{155}$$

Note also the identities

$$\hat{\kappa}^{\cdot\cdot\chi}_{\alpha\beta} = \tilde{F}^{-1A}_{\cdot\alpha}\tilde{F}^{-1B}_{\cdot\beta}\partial_{[A}\tilde{F}^\chi_{\cdot B]} = \tilde{F}^{-1A}_{\cdot[\alpha}\tilde{F}^{-1B}_{\cdot\beta]}\partial_A\tilde{F}^\chi_{\cdot B} = -\tilde{F}^{-1A}_{\cdot[\alpha}\tilde{F}^\chi_{\cdot|B|}\partial_A\tilde{F}^{-1B}_{\cdot\beta]} = -\tilde{F}^\chi_{\cdot B}\partial_{[\alpha}\tilde{F}^{-1B}_{\cdot\beta]}. \tag{156}$$

When $\tilde{\mathbf{F}}(X,t)$ is an integrable function of X^A and hence (108) applies, then $\hat{\kappa}^{\cdot\cdot\alpha}_{\beta\chi} = 0$. Assuming now that $\tilde{\mathbf{F}}$ is at least twice differentiable with respect to reference coordinates X^A, Riemann–Christoffel curvature tensor components $R^{\cdot\cdot\cdot A}_{BCD}$ of (33) in configuration B_0 map as [14, 27]

$$\hat{R}^{\cdot\cdot\cdot\alpha}_{\beta\chi\delta} = \tilde{F}^\alpha_{\cdot A}\tilde{F}^{-1B}_{\cdot\beta}\tilde{F}^{-1C}_{\cdot\chi}\tilde{F}^{-1D}_{\cdot\delta}R^{\cdot\cdot\cdot A}_{BCD} = 2\partial_{[\beta}\hat{\Gamma}^{\cdot\cdot\alpha}_{\chi]\delta} + 2\hat{\Gamma}^{\cdot\cdot\alpha}_{[\beta|\epsilon|}\hat{\Gamma}^{\cdot\cdot\epsilon}_{\chi]\delta} + 2\hat{\kappa}^{\cdot\cdot\epsilon}_{\beta\chi}\hat{\Gamma}^{\cdot\cdot\alpha}_{\epsilon\delta} = \hat{R}^{\cdot\cdot\cdot\alpha}_{[\beta\chi]\delta}. \tag{157}$$

Relations (154) and (157) are consistent with (34) and (35) [27]:

$$\nabla_{[\beta}\nabla_{\chi]}V^\alpha = \nabla_{[\beta}(\partial_{\chi]}V^\alpha + \hat{\Gamma}^{\cdot\cdot\alpha}_{\chi]\delta}V^\delta) = \frac{1}{2}\hat{R}^{\cdot\cdot\cdot\alpha}_{\beta\chi\delta}V^\delta - \hat{T}^{\cdot\cdot\delta}_{\beta\chi}\nabla_\delta V^\alpha, \tag{158}$$

$$\nabla_{[\beta}\nabla_{\chi]}\alpha_\delta = \nabla_{[\beta}(\partial_{\chi]}\alpha_\delta - \hat{\Gamma}^{\cdot\cdot\alpha}_{\chi]\delta}\alpha_\alpha) = -\frac{1}{2}\hat{R}^{\cdot\cdot\cdot\alpha}_{\beta\chi\delta}\alpha_\alpha - \hat{T}^{\cdot\cdot\alpha}_{\beta\chi}\nabla_\alpha\alpha_\delta, \tag{159}$$

where V^α and α_δ are respective anholonomic components of twice-differentiable vector and covector fields, and covariant differentiation with respect to anholonomic coordinates is defined formally later in (179). Skew components of second partial derivatives in (118) can be written as follows in terms of the anholonomic object:

$$\partial_{[\alpha}[\partial_{\beta]}(\cdot)] = \partial_A(\cdot)\partial_{[\alpha}\tilde{F}^{-1A}_{\ \ \cdot\beta]} = \partial_\chi(\cdot)\tilde{F}^{-1B}_{\ \ \cdot[\beta}\tilde{F}^{-1A}_{\ \ \cdot\alpha]}\partial_B\tilde{F}^\chi_{\cdot A} = -\hat{\kappa}_{\alpha\beta}^{\cdot\cdot\chi}\partial_\chi(\cdot). \tag{160}$$

Relationships analogous to (153)–(157) can be used to map connection coefficients and torsion and curvature tensors referred to spatial coordinates to intermediate space $\tilde{B}$ by replacing referential indices with spatial indices (e.g. $X^A \to x^a$) and replacing components $\tilde{F}^\alpha_{\cdot A}$ with $\bar{F}^{-1\alpha}_{\ \ \cdot a}$, the latter assumed twice differentiable with respect to x^a.

A convected coordinate representation of $\tilde{\mathbf{F}}(X,t)$ can be used to verify transformation (153) when $\Gamma_{BC}^{\cdot\cdot A} = \overset{G}{\Gamma}{}_{BC}^{\cdot\cdot A}$ corresponds to the Levi-Civita connection of (42) for Euclidean space on B_0. Convected anholonomic basis vectors and their reciprocals are defined as

$$\tilde{\mathbf{g}}'_\alpha(X,t) = \tilde{F}^{-1A}_{\ \ \cdot\alpha}\mathbf{G}_A, \qquad \tilde{\mathbf{g}}'^\alpha(X,t) = \tilde{F}^\alpha_{\cdot A}\mathbf{G}^A; \tag{161}$$

$$\langle\tilde{\mathbf{g}}'_\alpha, \tilde{\mathbf{g}}'^\beta\rangle = \tilde{F}^{-1A}_{\ \ \cdot\alpha}\tilde{F}^\beta_{\cdot B}\langle\mathbf{G}_A, \mathbf{G}^B\rangle = \tilde{F}^{-1A}_{\ \ \cdot\alpha}\tilde{F}^\beta_{\cdot B}\delta^B_A = \tilde{F}^{-1A}_{\ \ \cdot\alpha}\tilde{F}^\beta_{\cdot A} = \delta^\beta_\alpha. \tag{162}$$

The deformation map of the second of (102) can then be expressed as

$$\tilde{\mathbf{F}}(X,t) = \tilde{F}^\alpha_{\cdot A}\tilde{\mathbf{g}}_\alpha \otimes \mathbf{G}^A = \tilde{\mathbf{g}}_\alpha \otimes \tilde{F}^\alpha_{\cdot A}\mathbf{G}^A = \tilde{\mathbf{g}}_\alpha \otimes \tilde{\mathbf{g}}'^\alpha. \tag{163}$$

Components of the metric tensor corresponding to basis vectors in (161) are

$$\tilde{\mathbf{g}}'_\alpha \cdot \tilde{\mathbf{g}}'_\beta = \tilde{F}^{-1A}_{\ \ \cdot\alpha}\mathbf{G}_A \cdot \tilde{F}^{-1B}_{\ \ \cdot\beta}\mathbf{G}_B = \tilde{F}^{-1A}_{\ \ \cdot\alpha}\tilde{F}^{-1B}_{\ \ \cdot\beta}\mathbf{G}_A \cdot \mathbf{G}_B = \tilde{F}^{-1A}_{\ \ \cdot\alpha}G_{AB}\tilde{F}^{-1B}_{\ \ \cdot\beta}. \tag{164}$$

Taking the partial anholonomic derivative of $\tilde{\mathbf{g}}'_\alpha$ using (110) results in

$$\partial_\beta\tilde{\mathbf{g}}'_\alpha = \partial_\beta(\tilde{F}^{-1A}_{\ \ \cdot\alpha}\mathbf{G}_A) = \left(\tilde{F}^\chi_{\cdot C}\tilde{F}^{-1B}_{\ \ \cdot\beta}\tilde{F}^{-1A}_{\ \ \cdot\alpha}\overset{G}{\Gamma}{}_{BA}^{\cdot\cdot C} - \tilde{F}^{-1B}_{\ \ \cdot\beta}\tilde{F}^{-1A}_{\ \ \cdot\alpha}\partial_B\tilde{F}^\chi_{\cdot A}\right)\tilde{\mathbf{g}}'_\chi = \tilde{\Gamma}_{\beta\alpha}^{\cdot\cdot\chi}\tilde{\mathbf{g}}'_\chi, \tag{165}$$

in agreement with (153). The torsion tensor of (154) vanishes identically in this case from (42):

$$\tilde{T}_{\beta\chi}^{\cdot\cdot\alpha} = \tilde{F}^\alpha_{\cdot A}\tilde{F}^{-1B}_{\ \ \cdot\beta}\tilde{F}^{-1C}_{\ \ \cdot\chi}\overset{G}{\Gamma}{}_{[BC]}^{\cdot\cdot A} = \tilde{\Gamma}_{[\beta\chi]}^{\cdot\cdot\alpha} + \tilde{\kappa}_{\beta\chi}^{\cdot\cdot\alpha} = 0, \tag{166}$$

where the anholonomic object is the same as that of (155):

$$\tilde{\kappa}_{\beta\chi}^{\cdot\cdot\alpha} = \tilde{F}^{-1A}_{\ \ \cdot\beta}\tilde{F}^{-1B}_{\ \ \cdot\chi}\partial_{[A}\tilde{F}^\alpha_{\cdot B]}. \tag{167}$$

Following an analogous approach with $\bar{\mathbf{F}}^{-1}(x,t)$, convected anholonomic basis vectors and their reciprocals are

$$\bar{\mathbf{g}}'_\alpha(x,t) = \bar{F}^a_{\cdot\alpha}\mathbf{g}_a, \qquad \bar{\mathbf{g}}'^\alpha(x,t) = \bar{F}^{-1\alpha}_{\ \ \cdot a}\mathbf{g}^a. \tag{168}$$

$$\langle\bar{\mathbf{g}}'_\alpha, \bar{\mathbf{g}}'^\beta\rangle = \bar{F}^a_{\cdot\alpha}\bar{F}^{-1\beta}_{\ \ \cdot b}\langle\mathbf{g}_a, \mathbf{g}^b\rangle = \bar{F}^a_{\cdot\alpha}\bar{F}^{-1\beta}_{\ \ \cdot b}\delta^b_a = \bar{F}^a_{\cdot\alpha}\bar{F}^{-1\beta}_{\ \ \cdot a} = \delta^\beta_\alpha; \tag{169}$$

The deformation map of the first of (102) can be expressed as

$$\bar{\mathbf{F}}(X,t) = \bar{F}^a_{\cdot\alpha}\mathbf{g}_a \otimes \tilde{\mathbf{g}}^\alpha = \bar{\mathbf{g}}'_\alpha \otimes \tilde{\mathbf{g}}^\alpha. \tag{170}$$

Components of the metric tensor corresponding to basis vectors in (168) are

$$\bar{\mathbf{g}}'_\alpha \cdot \bar{\mathbf{g}}'_\beta = \bar{F}^a_{\cdot\alpha}\mathbf{g}_a \cdot \bar{F}^b_{\cdot\beta}\mathbf{g}_b = \bar{F}^a_{\cdot\alpha}\bar{F}^b_{\cdot\beta}\mathbf{g}_a \cdot \mathbf{g}_b = \bar{F}^a_{\cdot\alpha}g_{ab}\bar{F}^b_{\cdot\beta}. \tag{171}$$

Taking the partial anholonomic derivative of $\bar{\mathbf{g}}'_\alpha$ using (114) results in

$$\partial_\beta\bar{\mathbf{g}}'_\alpha = \partial_\beta(\bar{F}^a_{\cdot\alpha}\mathbf{g}_a) = \left(\bar{F}^{-1\chi}_{\ \ \cdot c}\bar{F}^b_{\cdot\beta}\bar{F}^a_{\cdot\alpha}\overset{g}{\Gamma}{}_{ba}^{\cdot\cdot c} - \bar{F}^b_{\cdot\beta}\bar{F}^a_{\cdot\alpha}\partial_b\bar{F}^{-1\chi}_{\ \ \cdot a}\right)\bar{\mathbf{g}}'_\chi = \bar{\Gamma}_{\beta\alpha}^{\cdot\cdot\chi}\bar{\mathbf{g}}'_\chi. \tag{172}$$

The torsion tensor of (154) vanishes identically in this case from (53):

$$\bar{T}_{\beta\chi}^{\cdot\cdot\alpha} = \bar{F}^{-1\alpha}_{\cdot a}\bar{F}^{b}_{\cdot\beta}\bar{F}^{c}_{\cdot\chi}\overset{g}{\Gamma}{}^{\cdot\cdot a}_{[bc]} = \bar{\Gamma}^{\cdot\cdot\alpha}_{[\beta\chi]} + \bar{\kappa}^{\cdot\cdot\alpha}_{\beta\chi} = 0, \tag{173}$$

where the corresponding anholonomic object is

$$\bar{\kappa}^{\cdot\cdot\alpha}_{\beta\chi} = \bar{F}^{a}_{\cdot\beta}\bar{F}^{b}_{\cdot\chi}\partial_{[a}\bar{F}^{-1\alpha}_{\cdot b]}. \tag{174}$$

Connection coefficients $\tilde{\Gamma}^{\cdot\cdot\chi}_{\beta\alpha}$ of (165) and $\bar{\Gamma}^{\cdot\cdot\chi}_{\beta\alpha}$ of (172) are generally different. However, following from decomposition (101), it can be shown that their skew covariant components are equal [14, p. 651]:

$$\bar{\Gamma}^{\cdot\cdot\chi}_{[\beta\alpha]} = -\bar{F}^{b}_{\cdot\beta}\bar{F}^{a}_{\cdot\alpha}\partial_{[b}\bar{F}^{-1\chi}_{\cdot a]} = -\tilde{F}^{-1B}_{\cdot\beta}\tilde{F}^{-1A}_{\cdot\alpha}\partial_{[B}\tilde{F}^{\chi}_{\cdot A]} = \tilde{\Gamma}^{\cdot\cdot\chi}_{[\beta\alpha]}. \tag{175}$$

Thus, anholonomic objects of each connection defined in (167) and (174) are equal:

$$\bar{\kappa}^{\cdot\cdot\chi}_{\beta\alpha} = -\bar{\Gamma}^{\cdot\cdot\chi}_{[\beta\alpha]} = -\tilde{\Gamma}^{\cdot\cdot\chi}_{[\beta\alpha]} = \tilde{\kappa}^{\cdot\cdot\chi}_{\beta\alpha}. \tag{176}$$

Computing skew partial derivatives of convected basis vectors as

$$\partial_{[\alpha}\tilde{\mathbf{g}}'_{\beta]} = \tilde{\Gamma}^{\cdot\cdot\chi}_{[\alpha\beta]}\tilde{\mathbf{g}}'_{\chi} = \tilde{\kappa}^{\cdot\cdot\chi}_{\beta\alpha}\tilde{\mathbf{g}}'_{\chi}, \qquad \partial_{[\alpha}\bar{\mathbf{g}}'_{\beta]} = \bar{\Gamma}^{\cdot\cdot\chi}_{[\alpha\beta]}\bar{\mathbf{g}}'_{\chi} = \bar{\kappa}^{\cdot\cdot\chi}_{\beta\alpha}\bar{\mathbf{g}}'_{\chi}, \tag{177}$$

the following local integrability conditions are equivalent:

$$\partial_{[\alpha}\tilde{\mathbf{g}}'_{\beta]} = 0 \Leftrightarrow \partial_{[\alpha}\bar{\mathbf{g}}'_{\beta]} = 0 \Leftrightarrow \tilde{\kappa}^{\cdot\cdot\chi}_{\beta\alpha} = \bar{\kappa}^{\cdot\cdot\chi}_{\beta\alpha} = 0 \Leftrightarrow \tilde{F}^{\alpha}_{\cdot A} = \partial_A\tilde{x}^{\alpha} \Leftrightarrow \bar{F}^{-1\alpha}_{\cdot a} = \partial_a\tilde{x}^{\alpha}. \tag{178}$$

3.2. Anholonomic covariant derivatives

Differentiation with respect to general anholonomic coordinates is developed. Covariant derivatives and corresponding connection coefficients are defined. Various choices of anholonomic connection coefficients corresponding to different basis vectors in the intermediate configuration are examined. Total covariant derivatives of two-point (and three-point) tensors with one or more indices referred to anholonomic space are defined. Divergence, curl, and Laplacian operations and corresponding identities are presented.

3.2.1. Differentiation. Covariant differentiation with respect to anholonomic coordinates is defined similarly to (31). Let $\mathbf{A}$ be a vector or higher-order tensor field with components $A^{\alpha\ldots\phi}_{\gamma\ldots\mu}$. The covariant derivative of $\mathbf{A}$ is computed as

$$\nabla_\nu A^{\alpha\ldots\phi}_{\gamma\ldots\mu} = \partial_\nu A^{\alpha\ldots\phi}_{\gamma\ldots\mu} + \Gamma^{\cdot\cdot\alpha}_{\nu\rho}A^{\rho\ldots\phi}_{\gamma\ldots\mu} + \cdots + \Gamma^{\cdot\cdot\phi}_{\nu\rho}A^{\alpha\ldots\rho}_{\gamma\ldots\mu} - \Gamma^{\cdot\cdot\rho}_{\nu\gamma}A^{\alpha\ldots\phi}_{\rho\ldots\mu} - \cdots - \Gamma^{\cdot\cdot\rho}_{\nu\mu}A^{\alpha\ldots\phi}_{\gamma\ldots\rho}. \tag{179}$$

Partial differentiation obeys (115). Connection coefficients referred to intermediate space $\tilde{B}$, written as $\Gamma^{\cdot\cdot\alpha}_{\nu\rho}$, in general consist of up to n^3 entries, where n is the dimensionality of Euclidean spaces B_0 and B. Partial derivatives of intermediate basis vectors and their reciprocals obey (restricting attention to metric connections), by definition,

$$\partial_\beta\tilde{\mathbf{g}}_\alpha = \Gamma^{\cdot\cdot\chi}_{\beta\alpha}\tilde{\mathbf{g}}_\chi, \qquad \partial_\beta\tilde{\mathbf{g}}^\alpha = -\Gamma^{\cdot\cdot\alpha}_{\beta\chi}\tilde{\mathbf{g}}^\chi. \tag{180}$$

Therefore, covariant derivatives of intermediate basis vectors vanish identically:

$$\nabla_\beta\tilde{\mathbf{g}}_\alpha = \partial_\beta\tilde{\mathbf{g}}_\alpha - \Gamma^{\cdot\cdot\chi}_{\beta\alpha}\tilde{\mathbf{g}}_\chi = 0, \qquad \nabla_\beta\tilde{\mathbf{g}}^\alpha = \partial_\beta\tilde{\mathbf{g}}^\alpha + \Gamma^{\cdot\cdot\alpha}_{\beta\chi}\tilde{\mathbf{g}}^\chi = 0. \tag{181}$$

For example, applying (179)–(180) to generic vector field $\mathbf{A} = A^\alpha\tilde{\mathbf{g}}_\alpha$ gives the gradient of $\mathbf{A}$:

$$\partial_\beta\mathbf{A}\otimes\tilde{\mathbf{g}}^\beta = \partial_\beta(A^\alpha\tilde{\mathbf{g}}_\alpha)\otimes\tilde{\mathbf{g}}^\beta = (\partial_\beta A^\alpha + A^\chi\Gamma^{\cdot\cdot\alpha}_{\beta\chi})\tilde{\mathbf{g}}_\alpha\otimes\tilde{\mathbf{g}}^\beta = \nabla_\beta A^\alpha\tilde{\mathbf{g}}_\alpha\otimes\tilde{\mathbf{g}}^\beta. \tag{182}$$

By definition, the gradient of a scalar field is the same as its partial derivative, e.g.

$$\nabla_\alpha A = \partial_\alpha A = \tilde{F}^{-1B}_{\cdot\alpha}\partial_B A = \bar{F}^{b}_{\cdot\alpha}\partial_b A. \tag{183}$$

Recall from (176) that the anholonomic object, denoted generically by $\kappa_{\beta\alpha}^{\cdot\cdot\chi} = \tilde{\kappa}_{\beta\alpha}^{\cdot\cdot\chi} = \bar{\kappa}_{\beta\alpha}^{\cdot\cdot\chi}$, obeys

$$\kappa_{\beta\alpha}^{\cdot\cdot\chi} = \tilde{F}^{-1B}{}_{\cdot\beta}\tilde{F}^{-1A}{}_{\cdot\alpha}\partial_{[B}\tilde{F}^{\chi}_{\cdot A]} = -\tilde{F}^{\chi}_{\cdot A}\partial_{[\beta}\tilde{F}^{-1A}{}_{\cdot\alpha]} = \bar{F}^{b}_{\cdot\beta}\bar{F}^{a}_{\cdot\alpha}\partial_{[b}\bar{F}^{-1\chi}{}_{\cdot a]} = -\bar{F}^{-1\chi}{}_{\cdot a}\partial_{[\beta}\bar{F}^{a}_{\cdot\alpha]} = \kappa_{[\beta\alpha]}^{\cdot\cdot\chi}. \tag{184}$$

The intermediate torsion, denoted generically by $T_{\beta\alpha}^{\cdot\cdot\chi}$, is defined using the final two equalities in (154):

$$T_{\beta\chi}^{\cdot\cdot\alpha} = \Gamma_{[\beta\chi]}^{\cdot\cdot\alpha} + \kappa_{\beta\chi}^{\cdot\cdot\alpha} = T_{[\beta\chi]}^{\cdot\cdot\alpha}, \tag{185}$$

Here, the first two equalities in (154) are not required to hold for this generic definition of the torsion of an anholonomic space (i.e. the torsion in (185) need not map between configurations as a true tensor). Similarly, the intermediate curvature, denoted generically by $R_{\beta\chi\delta}^{\cdot\cdot\cdot\alpha}$, is defined using only the final two equalities in (157):

$$R_{\beta\chi\delta}^{\cdot\cdot\cdot\alpha} = 2\partial_{[\beta}\Gamma_{\chi]\delta}^{\cdot\cdot\alpha} + 2\Gamma_{[\beta|\epsilon|}^{\cdot\cdot\alpha}\Gamma_{\chi]\delta}^{\cdot\cdot\epsilon} + 2\kappa_{\beta\chi}^{\cdot\cdot\epsilon}\Gamma_{\epsilon\delta}^{\cdot\cdot\alpha} = R_{[\beta\chi]\delta}^{\cdot\cdot\cdot\alpha}. \tag{186}$$

Here, the first two equalities in (157) are not required to hold for this generic definition of the curvature of an anholonomic space (i.e. the curvature in (186) need not map between configurations as a true tensor).

With definitions (184), (185), and (186) now given, skew gradients can be obtained. Specifically, from (160),

$$\partial_{[\alpha}[\partial_{\beta]}(\cdot)] = -\kappa_{\alpha\beta}^{\cdot\cdot\chi}\partial_{\chi}(\cdot). \tag{187}$$

For the twice-differentiable scalar field A,

$$\nabla_{[\alpha}(\nabla_{\beta]}A) = \partial_{[\alpha}(\partial_{\beta]}A) - \Gamma_{[\alpha\beta]}^{\cdot\cdot\chi}\partial_{\chi}A = -(\kappa_{\alpha\beta}^{\cdot\cdot\chi} + \Gamma_{[\alpha\beta]}^{\cdot\cdot\chi})\partial_{\chi}A = -T_{\alpha\beta}^{\cdot\cdot\chi}\partial_{\chi}A. \tag{188}$$

It can be shown that (158) and (159) also hold, i.e.

$$\nabla_{[\beta}\nabla_{\chi]}V^{\alpha} = \frac{1}{2}R_{\beta\chi\delta}^{\cdot\cdot\cdot\alpha}V^{\delta} - T_{\beta\chi}^{\cdot\cdot\delta}\nabla_{\delta}V^{\alpha}, \qquad \nabla_{[\beta}\nabla_{\chi]}\alpha_{\delta} = -\frac{1}{2}R_{\beta\chi\delta}^{\cdot\cdot\cdot\alpha}\alpha_{\alpha} - T_{\beta\chi}^{\cdot\cdot\alpha}\nabla_{\alpha}\alpha_{\delta}, \tag{189}$$

where V^{α} and α_{δ} denote components of twice-differentiable vector and covector fields, respectively.

3.2.2. Anholonomic connection coefficients. Particular choices of coefficients $\Gamma_{\beta\chi}^{\cdot\cdot\alpha}$ entering (179) and (180) are discussed in what follows.

First consider the case when (108) applies, such that $\tilde{B}$ can be regarded as a Euclidean space and a position vector $\tilde{\mathbf{x}}$ can be assigned to any point $\tilde{x}(X,t) \in \tilde{B}$. In that case, basis vectors and metric tensor components can be defined as in (138) and (139), and any time-independent Euclidean coordinate system (e.g. fixed curvilinear or Cartesian coordinates) can be used for $\tilde{x}^{\alpha}$. Coefficients of the Levi-Civita connection on $\tilde{B}$ are, analogously to (42),

$$\Gamma_{\beta\chi}^{\cdot\cdot\alpha}(\tilde{x}) = \overset{\tilde{g}}{\Gamma}{}_{\beta\chi}^{\cdot\cdot\alpha} = \frac{1}{2}\tilde{g}^{\alpha\delta}(\partial_{\beta}\tilde{g}_{\chi\delta} + \partial_{\chi}\tilde{g}_{\beta\delta} - \partial_{\delta}\tilde{g}_{\beta\chi}) = \overset{\tilde{g}}{\Gamma}{}_{\chi\beta}^{\cdot\cdot\alpha}. \tag{190}$$

Since $\tilde{x}^{\alpha}$ are holonomic coordinates in Euclidean space, the torsion, anholonomic object, and curvature associated with coefficients (190) all vanish identically. Skew gradients of basis vectors also vanish [28]: $\partial_{[\alpha}\tilde{\mathbf{g}}_{\beta]} = \partial_{[\alpha}\partial_{\beta]}\tilde{\mathbf{x}} = 0$.

Next consider the case when (109) applies, such that single-valued coordinates $\tilde{x}^{\alpha}(X,t)$ continuously differentiable with respect to X^{A} do not exist in $\tilde{B}$. When, following (140), identical coordinate systems are used in configurations B_0 and $\tilde{B}$:

$$\tilde{\mathbf{g}}_{\alpha}(X) = \delta_{\alpha}^{A}\mathbf{G}_{A}, \tag{191}$$

$$\tilde{\mathbf{g}}^{\alpha}(X) = \tilde{g}^{\alpha\beta}\tilde{\mathbf{g}}_{\beta} = \delta_{C}^{\alpha}\delta_{D}^{\beta}G^{CD}\delta_{\beta}^{A}\mathbf{G}_{A} = \delta_{C}^{\alpha}G^{CA}\mathbf{G}_{A} = \delta_{A}^{\alpha}\mathbf{G}^{A}, \tag{192}$$

$$\tilde{g}_{\alpha\beta}(X) = \delta_{\alpha}^{A}\delta_{\beta}^{B}G_{AB}, \quad \tilde{g}^{\alpha\beta}(X) = \delta_{A}^{\alpha}\delta_{B}^{\beta}G^{AB}. \tag{193}$$

Taking the partial derivative of (191) and applying (45) and (110) gives

$$\partial_\beta \tilde{\mathbf{g}}_\alpha = \delta^A_\alpha \partial_\beta \mathbf{G}_A = \delta^A_\alpha \tilde{F}^{-1B}_{\cdot\beta} \partial_B \mathbf{G}_A = \delta^A_\alpha \tilde{F}^{-1B}_{\cdot\beta} \overset{G}{\Gamma}{}^{\cdot\cdot C}_{BA} \mathbf{G}_C = \delta^A_\alpha \tilde{F}^{-1B}_{\cdot\beta} \overset{G}{\Gamma}{}^{\cdot\cdot C}_{BA} \delta^\chi_C \tilde{\mathbf{g}}_\chi = \Gamma^{\cdot\cdot\chi}_{\beta\alpha} \tilde{\mathbf{g}}_\chi. \tag{194}$$

Similarly, taking the partial derivative of (192) and applying (46) and (110) gives

$$\partial_\beta \tilde{\mathbf{g}}^\alpha = \delta^\alpha_A \partial_\beta \mathbf{G}^A = \delta^\alpha_A \tilde{F}^{-1B}_{\cdot\beta} \partial_B \mathbf{G}^A = -\delta^\alpha_A \tilde{F}^{-1B}_{\cdot\beta} \overset{G}{\Gamma}{}^{\cdot\cdot A}_{BC} \mathbf{G}^C = -\delta^\alpha_A \tilde{F}^{-1B}_{\cdot\beta} \overset{G}{\Gamma}{}^{\cdot\cdot A}_{BC} \delta^C_\chi \tilde{\mathbf{g}}^\chi = -\Gamma^{\cdot\cdot\alpha}_{\beta\chi} \tilde{\mathbf{g}}^\chi. \tag{195}$$

Comparing (180), (194), and (195), connection coefficients consistent with (140) are time dependent:

$$\Gamma^{\cdot\cdot\alpha}_{\beta\chi}(X,t) = \delta^\alpha_A \delta^C_\chi \overset{G}{\Gamma}{}^{\cdot\cdot A}_{BC}(X) \tilde{F}^{-1B}_{\cdot\beta}(X,t). \tag{196}$$

Since Levi-Civita connection coefficients $\overset{G}{\Gamma}{}^{\cdot\cdot A}_{BC}$ are symmetric, it follows that $\Gamma^{\cdot\cdot\alpha}_{\beta\chi}$ defined in (196) obeys

$$\Gamma^{\cdot\cdot\alpha}_{\beta\chi} = \delta^\alpha_A \delta^C_\chi \overset{G}{\Gamma}{}^{\cdot\cdot A}_{BC} \tilde{F}^{-1B}_{\cdot\beta} = \delta^\alpha_A \delta^C_\chi \overset{G}{\Gamma}{}^{\cdot\cdot A}_{CB} \tilde{F}^{-1B}_{\cdot\beta} = \delta^\alpha_A \delta^B_\chi \overset{G}{\Gamma}{}^{\cdot\cdot A}_{CB} \tilde{F}^{-1C}_{\cdot\beta}. \tag{197}$$

Covariant indices of $\Gamma^{\cdot\cdot\alpha}_{\beta\chi}$ are generally not symmetric; the left covariant component corresponding to differentiation by $\nabla_\beta(\cdot)$ in (179) correlates with $\tilde{F}^{-1C}_{\cdot\beta}$. The torsion of (196) is defined as in (185):

$$T^{\cdot\cdot\alpha}_{\beta\chi} = \Gamma^{\cdot\cdot\alpha}_{[\beta\chi]} + \tilde{\kappa}^{\cdot\cdot\alpha}_{\beta\chi} = \delta^\alpha_A \overset{G}{\Gamma}{}^{\cdot\cdot A}_{BC} \tilde{F}^{-1B}_{\cdot[\beta} \delta^C_{\chi]} + \tilde{F}^{-1A}_{\cdot[\beta} \tilde{F}^{-1B}_{\cdot\chi]} \partial_A \tilde{F}^\alpha_{\cdot B}, \tag{198}$$

where $\tilde{\kappa}^{\cdot\cdot\alpha}_{\beta\chi}$ is the anholonomic object associated with $\tilde{\mathbf{F}}$ of (167). When the anholonomic object $\tilde{\kappa}^{\cdot\cdot\alpha}_{\beta\chi}$ is non-zero, skew partial derivatives as in (187) are generally non-zero. When the torsion $T^{\cdot\cdot\alpha}_{\beta\chi}$ is non-zero, skew covariant derivatives of a scalar field as in (188) are generally non-zero. The Riemann–Christoffel curvature associated with (196) is defined as in (186):

$$R^{\cdot\cdot\cdot\alpha}_{\beta\chi\delta} = 2\partial_{[\beta}\Gamma^{\cdot\cdot\alpha}_{\chi]\delta} + 2\Gamma^{\cdot\cdot\alpha}_{[\beta|\epsilon|}\Gamma^{\cdot\cdot\epsilon}_{\chi]\delta} + 2\tilde{\kappa}^{\cdot\cdot\epsilon}_{\beta\chi}\Gamma^{\cdot\cdot\alpha}_{\epsilon\delta} = -2\delta^\alpha_A \delta^D_\delta \tilde{F}^{-1B}_{\cdot[\beta} \tilde{F}^{-1C}_{\cdot\chi]} \partial_B \overset{G}{\Gamma}{}^{\cdot\cdot A}_{CD}, \tag{199}$$

where the vanishing of the curvature tensor of Euclidean reference space (43) has been used. When curvature $R^{\cdot\cdot\cdot\alpha}_{\beta\chi\delta}$ of (199) and torsion $T^{\cdot\cdot\alpha}_{\beta\chi}$ of (198) do not vanish, skew covariant derivatives of vector and covector fields are generally non-zero, as in (189). It is noted that torsion and curvature defined in this way can each be non-zero even when configuration $\tilde{B}$ is holonomic, i.e. even when $\tilde{F}^\alpha_{\cdot A} = \partial_A \tilde{x}^\alpha$ and $\tilde{\kappa}^{\cdot\cdot\alpha}_{\beta\chi} = 0$. Using (49), the (negative) covariant derivative of the metric tensor is

$$M_{\alpha\beta\chi} = -\nabla_\alpha \tilde{g}_{\beta\chi} = -\partial_\alpha \tilde{g}_{\beta\chi} + \Gamma^{\cdot\cdot\delta}_{\alpha\beta}\tilde{g}_{\delta\chi} + \Gamma^{\cdot\cdot\delta}_{\alpha\chi}\tilde{g}_{\beta\delta} = -\delta^B_\beta \delta^C_\chi \tilde{F}^{-1A}_{\cdot\alpha} G_{BC;A} = 0. \tag{200}$$

Therefore, anholonomic covariant differentiation commutes with lowering indices via the metric $\tilde{g}_{\alpha\beta} = \delta^A_\alpha \delta^B_\beta G_{AB}$.

Now consider the case when (113) applies, such that single-valued coordinates $\tilde{x}^\alpha(x,t)$ continuously differentiable with respect to x^a do not exist in $\tilde{B}$. Following (146), identical coordinate systems are used in configurations B and $\tilde{B}$:

$$\tilde{\mathbf{g}}_\alpha(x) = \delta^a_\alpha \mathbf{g}_a, \tag{201}$$

$$\tilde{\mathbf{g}}^\alpha(x) = \tilde{g}^{\alpha\beta}\tilde{\mathbf{g}}_\beta = \delta^\alpha_c \delta^\beta_d g^{cd} \delta^a_\beta \mathbf{g}_a = \delta^\alpha_c g^{ca} \mathbf{g}_a = \delta^\alpha_a \mathbf{g}^a, \tag{202}$$

$$\tilde{g}_{\alpha\beta}(x) = \delta^a_\alpha \delta^b_\beta g_{ab}, \qquad \tilde{g}^{\alpha\beta}(x) = \delta^\alpha_a \delta^\beta_b g^{ab}. \tag{203}$$

Taking the partial derivative of (201) and applying (56) and (114) gives

$$\partial_\beta \tilde{\mathbf{g}}_\alpha = \delta^a_\alpha \partial_\beta \mathbf{g}_a = \delta^a_\alpha \bar{F}^b_{\cdot\beta} \partial_b \mathbf{g}_a = \delta^a_\alpha \bar{F}^b_{\cdot\beta} \overset{g}{\Gamma}{}^{\cdot\cdot c}_{ba} \mathbf{g}_c = \delta^a_\alpha \bar{F}^b_{\cdot\beta} \overset{g}{\Gamma}{}^{\cdot\cdot c}_{ba} \delta^\chi_c \tilde{\mathbf{g}}_\chi = \Gamma^{\cdot\cdot\chi}_{\beta\alpha} \tilde{\mathbf{g}}_\chi. \tag{204}$$

Similarly, taking the partial derivative of (202) and applying (57) and (114) gives

$$\partial_\beta \tilde{\mathbf{g}}^\alpha = \delta^\alpha_a \partial_\beta \mathbf{g}^a = \delta^\alpha_a \bar{F}^b_{\cdot\beta} \partial_b \mathbf{g}^a = -\delta^\alpha_a \bar{F}^b_{\cdot\beta} \overset{g}{\Gamma}{}^{\cdot\cdot a}_{bc} \mathbf{g}^c = -\delta^\alpha_a \bar{F}^b_{\cdot\beta} \overset{g}{\Gamma}{}^{\cdot\cdot a}_{bc} \delta^c_\chi \tilde{\mathbf{g}}^\chi = -\Gamma^{\cdot\cdot\alpha}_{\beta\chi} \tilde{\mathbf{g}}^\chi. \tag{205}$$

Comparing (180), (204), and (205), connection coefficients consistent with (146) are time dependent:

$$\Gamma_{\beta\chi}^{\cdot\cdot\alpha}(x,t) = \delta_a^\alpha \delta_\chi^c \overset{g}{\Gamma}{}_{bc}^{\cdot\cdot a}(x) \bar{F}_{\cdot\beta}^b(x,t). \tag{206}$$

Since the Levi-Civita connection $\overset{g}{\Gamma}{}_{bc}^{\cdot\cdot a}$ is symmetric in covariant indices, it follows that $\Gamma_{\beta\chi}^{\cdot\cdot\alpha}$ of (206) obeys

$$\Gamma_{\beta\chi}^{\cdot\cdot\alpha} = \delta_a^\alpha \delta_\chi^c \overset{g}{\Gamma}{}_{bc}^{\cdot\cdot a} \bar{F}_{\cdot\beta}^b = \delta_a^\alpha \delta_\chi^c \overset{g}{\Gamma}{}_{cb}^{\cdot\cdot a} \bar{F}_{\cdot\beta}^b = \delta_a^\alpha \delta_\chi^b \overset{g}{\Gamma}{}_{cb}^{\cdot\cdot a} \bar{F}_{\cdot\beta}^c. \tag{207}$$

Covariant components of $\Gamma_{\beta\chi}^{\cdot\cdot\alpha}$ are generally not symmetric; here the left covariant component corresponding to differentiation by $\nabla_\beta(\cdot)$ in (179) correlates with $\bar{F}_{\cdot\beta}^c$. The torsion of (206) is defined as in (185):

$$T_{\beta\chi}^{\cdot\cdot\alpha} = \Gamma_{[\beta\chi]}^{\cdot\cdot\alpha} + \bar{\kappa}_{\beta\chi}^{\cdot\cdot\alpha} = \delta_a^\alpha \overset{g}{\Gamma}{}_{bc}^{\cdot\cdot a} \bar{F}_{\cdot[\beta}^b \delta_{\chi]}^c + \bar{F}_{\cdot[\beta}^a \bar{F}_{\cdot\chi]}^b \partial_a \bar{F}^{-1\alpha}_{\cdot b}, \tag{208}$$

where $\bar{\kappa}_{\beta\chi}^{\cdot\cdot\alpha}$ is the anholonomic object associated with $\bar{\mathbf{F}}$ of (174). When anholonomic object $\bar{\kappa}_{\beta\chi}^{\cdot\cdot\alpha}$ is non-zero, skew partial derivatives as in (187) are generally non-zero. When torsion $T_{\beta\chi}^{\cdot\cdot\alpha}$ is non-zero, skew covariant derivatives of a scalar field as in (188) are generally non-zero. The Riemann–Christoffel curvature of (206) is defined as in (186):

$$R_{\beta\chi\delta}^{\cdot\cdot\cdot\alpha} = 2\partial_{[\beta}\Gamma_{\chi]\delta}^{\cdot\cdot\alpha} + 2\Gamma_{[\beta\epsilon}^{\cdot\cdot\alpha}\Gamma_{\chi]\delta}^{\cdot\cdot\epsilon} + 2\bar{\kappa}_{\beta\chi}^{\cdot\cdot\epsilon}\Gamma_{\epsilon\delta}^{\cdot\cdot\alpha} = -2\delta_a^\alpha \delta_\delta^d \bar{F}_{\cdot[\beta}^b \bar{F}_{\cdot\chi]}^c \partial_b \overset{g}{\Gamma}{}_{cd}^{\cdot\cdot a}, \tag{209}$$

where the vanishing of the curvature tensor of Euclidean current space (54) has been used. When the curvature $R_{\beta\chi\delta}^{\cdot\cdot\cdot\alpha}$ of (209) and torsion $T_{\beta\chi}^{\cdot\cdot\alpha}$ of (208) do not vanish, skew covariant derivatives of vector and covector fields are generally non-zero, as in (189). Note that the curvature and torsion defined in this manner need not vanish even if configuration $\tilde{B}$ is holonomic, i.e. even if $\bar{F}^{-1\alpha}_{\cdot a} = \partial_a \tilde{x}^\alpha$ and anholonomic object $\bar{\kappa}_{\beta\chi}^{\cdot\cdot\alpha} = 0$. Connection coefficients and curvature for this case are worked out explicitly for the choice of cylindrical coordinates in the Appendix. The (negative) covariant derivative of the metric tensor is

$$M_{\alpha\beta\chi} = -\nabla_\alpha \tilde{g}_{\beta\chi} = -\partial_\alpha \tilde{g}_{\beta\chi} + \Gamma_{\alpha\beta}^{\cdot\cdot\delta}\tilde{g}_{\delta\chi} + \Gamma_{\alpha\chi}^{\cdot\cdot\delta}\tilde{g}_{\beta\delta} = -\delta_\beta^b \delta_\chi^c \bar{F}_{\cdot\alpha}^a g_{bc;a} = 0, \tag{210}$$

where (62) has been used. Thus, covariant differentiation commutes with lowering indices via $\tilde{g}_{\alpha\beta} = \delta_\alpha^a \delta_\beta^b g_{ab}$.

Finally consider the simplest case whereby Cartesian bases are used for $\tilde{B}$, as in (152):

$$\tilde{\mathbf{g}}_\alpha = \mathbf{e}_\alpha, \quad \tilde{\mathbf{g}}^\alpha = \mathbf{e}^\alpha, \quad \tilde{g}_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \delta_{\alpha\beta}, \quad \tilde{g}^{\alpha\beta} = \mathbf{e}^\alpha \cdot \mathbf{e}^\beta = \delta^{\alpha\beta}. \tag{211}$$

In this case, partial derivatives of basis vectors vanish identically,

$$\partial_\beta \tilde{\mathbf{g}}_\alpha = \partial_\beta \mathbf{e}_\alpha = \tilde{F}^{-1B}_{\cdot\beta}\partial_B \mathbf{e}_\alpha = \bar{F}_{\cdot\beta}^b \partial_b \mathbf{e}_\alpha = \Gamma_{\beta\alpha}^{\cdot\cdot\chi}\tilde{\mathbf{g}}_\chi = 0, \tag{212}$$

as do partial derivatives of their reciprocals,

$$\partial_\beta \tilde{\mathbf{g}}^\alpha = \partial_\beta \mathbf{e}^\alpha = \tilde{F}^{-1B}_{\cdot\beta}\partial_B \mathbf{e}^\alpha = \bar{F}_{\cdot\beta}^b \partial_b \mathbf{e}^\alpha = -\Gamma_{\beta\chi}^{\cdot\cdot\alpha}\tilde{\mathbf{g}}^\chi = 0. \tag{213}$$

It follows that connection coefficients on $\tilde{B}$ must also vanish in this case:

$$\Gamma_{\beta\chi}^{\cdot\cdot\alpha} = 0. \tag{214}$$

Note however that the anholonomic object need not vanish; hence, the torsion of (185) is

$$T_{\beta\chi}^{\cdot\cdot\alpha} = \kappa_{\beta\chi}^{\cdot\cdot\alpha}. \tag{215}$$

Curvature tensor (186) vanishes identically since $\Gamma_{\beta\chi}^{\cdot\cdot\alpha} = 0$. The covariant derivative of the metric also vanishes:

$$M_{\alpha\beta\chi} = -\nabla_\alpha \tilde{g}_{\beta\chi} = -\partial_\alpha \delta_{\beta\chi} = -\tilde{F}^{-1A}_{\cdot\alpha}\partial_A \delta_{\beta\chi} = -\bar{F}_{\cdot\alpha}^a \partial_a \delta_{\beta\chi} = 0. \tag{216}$$

3.2.3. Total covariant derivatives. Covariant differentiation of two-point tensor fields with one or more components referred to anholonomic coordinates (i.e. one or more indices referred to configuration $\tilde{B}$) is defined following arguments similar to those of Section 2.3.2. First consider a generic two-point tensor of the form $\mathbf{A}(X,t)$, with components $A^{\alpha\ldots\phi\ A\ldots F}_{\gamma\ldots\mu\ G\ldots M}$. The total covariant derivative of $\mathbf{A}$ is calculated as

$$\begin{aligned}
(A^{\alpha\ldots\phi\ A\ldots F}_{\gamma\ldots\mu\ G\ldots M}){:}_\nu = {} & \partial_\nu(A^{\alpha\ldots\phi\ A\ldots F}_{\gamma\ldots\mu\ G\ldots M}) \\
& + \Gamma^{\cdot\cdot\alpha}_{\nu\rho}A^{\rho\ldots\phi\ A\ldots F}_{\gamma\ldots\mu\ G\ldots M} + \cdots + \Gamma^{\cdot\cdot\phi}_{\nu\rho}A^{\alpha\ldots\rho\ A\ldots F}_{\gamma\ldots\mu\ G\ldots M} \\
& - \Gamma^{\cdot\cdot\rho}_{\nu\gamma}A^{\alpha\ldots\phi\ A\ldots F}_{\rho\ldots\mu\ G\ldots M} - \cdots - \Gamma^{\cdot\cdot\rho}_{\nu\mu}A^{\alpha\ldots\phi\ A\ldots F}_{\gamma\ldots\rho\ G\ldots M} \\
& + \tilde{F}^{-1N}_{\ \ \ \cdot\nu}\overset{G}{\Gamma}{}^{\cdot\cdot A}_{NR}A^{\alpha\ldots\phi\ R\ldots F}_{\gamma\ldots\mu\ G\ldots M} + \cdots + \tilde{F}^{-1N}_{\ \ \ \cdot\nu}\overset{G}{\Gamma}{}^{\cdot\cdot F}_{NR}A^{\alpha\ldots\phi\ A\ldots R}_{\gamma\ldots\mu\ G\ldots M} \\
& - \tilde{F}^{-1N}_{\ \ \ \cdot\nu}\overset{G}{\Gamma}{}^{\cdot\cdot R}_{NG}A^{\alpha\ldots\phi\ A\ldots F}_{\gamma\ldots\mu\ R\ldots M} - \cdots - \tilde{F}^{-1N}_{\ \ \ \cdot\nu}\overset{G}{\Gamma}{}^{\cdot\cdot R}_{NM}A^{\alpha\ldots\phi\ A\ldots F}_{\gamma\ldots\mu\ G\ldots R} \\
= {} & \big[\partial_N(A^{\alpha\ldots\phi\ A\ldots F}_{\gamma\ldots\mu\ G\ldots M}) \\
& + \overset{G}{\Gamma}{}^{\cdot\cdot A}_{NR}A^{\alpha\ldots\phi\ R\ldots F}_{\gamma\ldots\mu\ G\ldots M} + \cdots + \overset{G}{\Gamma}{}^{\cdot\cdot F}_{NR}A^{\alpha\ldots\phi\ A\ldots R}_{\gamma\ldots\mu\ G\ldots M} \\
& - \overset{G}{\Gamma}{}^{\cdot\cdot R}_{NG}A^{\alpha\ldots\phi\ A\ldots F}_{\gamma\ldots\mu\ R\ldots M} - \cdots - \overset{G}{\Gamma}{}^{\cdot\cdot R}_{NM}A^{\alpha\ldots\phi\ A\ldots F}_{\gamma\ldots\mu\ G\ldots R}\big]\tilde{F}^{-1N}_{\ \ \ \cdot\nu} \\
& + \Gamma^{\cdot\cdot\alpha}_{\nu\rho}A^{\rho\ldots\phi\ A\ldots F}_{\gamma\ldots\mu\ G\ldots M} + \cdots + \Gamma^{\cdot\cdot\phi}_{\nu\rho}A^{\alpha\ldots\rho\ A\ldots F}_{\gamma\ldots\mu\ G\ldots M} \\
& - \Gamma^{\cdot\cdot\rho}_{\nu\gamma}A^{\alpha\ldots\phi\ A\ldots F}_{\rho\ldots\mu\ G\ldots M} - \cdots - \Gamma^{\cdot\cdot\rho}_{\nu\mu}A^{\alpha\ldots\phi\ A\ldots F}_{\gamma\ldots\rho\ G\ldots M} \\
= {} & (A^{\alpha\ldots\phi\ A\ldots F}_{\gamma\ldots\mu\ G\ldots M});_N\tilde{F}^{-1N}_{\ \ \ \cdot\nu} \\
& + \Gamma^{\cdot\cdot\alpha}_{\nu\rho}A^{\rho\ldots\phi\ A\ldots F}_{\gamma\ldots\mu\ G\ldots M} + \cdots + \Gamma^{\cdot\cdot\phi}_{\nu\rho}A^{\alpha\ldots\rho\ A\ldots F}_{\gamma\ldots\mu\ G\ldots M} \\
& - \Gamma^{\cdot\cdot\rho}_{\nu\gamma}A^{\alpha\ldots\phi\ A\ldots F}_{\rho\ldots\mu\ G\ldots M} - \cdots - \Gamma^{\cdot\cdot\rho}_{\nu\mu}A^{\alpha\ldots\phi\ A\ldots F}_{\gamma\ldots\rho\ G\ldots M} \\
= {} & (A^{\alpha\ldots\phi\ A\ldots F}_{\gamma\ldots\mu\ G\ldots M}){:}_N\tilde{F}^{-1N}_{\ \ \ \cdot\nu}.
\end{aligned} \tag{217}$$

Next consider a generic two-point tensor of the form $\mathbf{A}(x,t)$, with components $A^{\alpha\ldots\phi\ a\ldots f}_{\gamma\ldots\mu\ g\ldots m}$. The total covariant derivative of $\mathbf{A}$ is computed analogously to (217), replacing referential coordinates X^N with spatial coordinates x^n and $\tilde{F}^{-1N}_{\ \ \ \cdot\nu}$ with $\bar{F}^{n}_{\cdot\nu}$:

$$\begin{aligned}
(A^{\alpha\ldots\phi\ a\ldots f}_{\gamma\ldots\mu\ g\ldots m}){:}_\nu = {} & \partial_\nu(A^{\alpha\ldots\phi\ a\ldots f}_{\gamma\ldots\mu\ g\ldots m}) \\
& + \Gamma^{\cdot\cdot\alpha}_{\nu\rho}A^{\rho\ldots\phi\ a\ldots f}_{\gamma\ldots\mu\ g\ldots m} + \cdots + \Gamma^{\cdot\cdot\phi}_{\nu\rho}A^{\alpha\ldots\rho\ a\ldots f}_{\gamma\ldots\mu\ g\ldots m} \\
& - \Gamma^{\cdot\cdot\rho}_{\nu\gamma}A^{\alpha\ldots\phi\ a\ldots f}_{\rho\ldots\mu\ g\ldots m} - \cdots - \Gamma^{\cdot\cdot\rho}_{\nu\mu}A^{\alpha\ldots\phi\ a\ldots f}_{\gamma\ldots\rho\ g\ldots m} \\
& + \bar{F}^{n}_{\cdot\nu}\overset{g}{\Gamma}{}^{\cdot\cdot a}_{nr}A^{\alpha\ldots\phi\ r\ldots f}_{\gamma\ldots\mu\ g\ldots m} + \cdots + \bar{F}^{n}_{\cdot\nu}\overset{g}{\Gamma}{}^{\cdot\cdot f}_{nr}A^{\alpha\ldots\phi\ a\ldots r}_{\gamma\ldots\mu\ g\ldots m} \\
& - \bar{F}^{n}_{\cdot\nu}\overset{g}{\Gamma}{}^{\cdot\cdot r}_{ng}A^{\alpha\ldots\phi\ a\ldots f}_{\gamma\ldots\mu\ r\ldots m} - \cdots - \bar{F}^{n}_{\cdot\nu}\overset{g}{\Gamma}{}^{\cdot\cdot r}_{nm}A^{\alpha\ldots\phi\ a\ldots f}_{\gamma\ldots\mu\ g\ldots r} \\
= {} & \big[\partial_n(A^{\alpha\ldots\phi\ a\ldots f}_{\gamma\ldots\mu\ g\ldots m}) \\
& + \overset{g}{\Gamma}{}^{\cdot\cdot a}_{nr}A^{\alpha\ldots\phi\ r\ldots f}_{\gamma\ldots\mu\ g\ldots m} + \cdots + \overset{g}{\Gamma}{}^{\cdot\cdot f}_{nr}A^{\alpha\ldots\phi\ a\ldots r}_{\gamma\ldots\mu\ g\ldots m} \\
& - \overset{g}{\Gamma}{}^{\cdot\cdot r}_{ng}A^{\alpha\ldots\phi\ a\ldots f}_{\gamma\ldots\mu\ r\ldots m} - \cdots - \overset{g}{\Gamma}{}^{\cdot\cdot r}_{nm}A^{\alpha\ldots\phi\ a\ldots g}_{\gamma\ldots\mu\ g\ldots r}\big]\bar{F}^{n}_{\cdot\nu} \\
& + \Gamma^{\cdot\cdot\alpha}_{\nu\rho}A^{\rho\ldots\phi\ a\ldots f}_{\gamma\ldots\mu\ g\ldots m} + \cdots + \Gamma^{\cdot\cdot\phi}_{\nu\rho}A^{\alpha\ldots\rho\ a\ldots f}_{\gamma\ldots\mu\ g\ldots m} \\
& - \Gamma^{\cdot\cdot\rho}_{\nu\gamma}A^{\alpha\ldots\phi\ a\ldots f}_{\rho\ldots\mu\ g\ldots m} - \cdots - \Gamma^{\cdot\cdot\rho}_{\nu\mu}A^{\alpha\ldots\phi\ a\ldots f}_{\gamma\ldots\rho\ g\ldots m}
\end{aligned} \tag{218}$$

$$
\begin{aligned}
&= (A^{\alpha\ldots\phi\ a\ldots f}_{\gamma\ldots\mu\ g\ldots m})_{;n}\bar{F}^{n}_{\cdot\nu}\\
&\quad + \Gamma^{\cdot\cdot\alpha}_{\nu\rho}A^{\rho\ldots\phi\ a\ldots f}_{\gamma\ldots\mu\ g\ldots m} + \cdots + \Gamma^{\cdot\cdot\phi}_{\nu\rho}A^{\alpha\ldots\rho\ a\ldots f}_{\gamma\ldots\mu\ g\ldots m}\\
&\quad - \Gamma^{\cdot\cdot\rho}_{\nu\gamma}A^{\alpha\ldots\phi\ a\ldots f}_{\rho\ldots\mu\ g\ldots m} - \cdots - \Gamma^{\cdot\cdot\rho}_{\nu\mu}A^{\alpha\ldots\phi\ a\ldots f}_{\gamma\ldots\rho\ g\ldots m}\\
&= (A^{\alpha\ldots\phi\ a\ldots f}_{\gamma\ldots\mu\ g\ldots m})_{:n}\bar{F}^{n}_{\cdot\nu}.
\end{aligned}
$$

For example, letting $\mathbf{A}(X,t) = \tilde{\mathbf{F}}(X,t) = \tilde{F}^{\alpha}_{\cdot A}\tilde{\mathbf{g}}_{\alpha}\otimes\mathbf{G}^{A}$, the material gradient is computed using (217) as

$$\overset{G}{\nabla}\tilde{\mathbf{F}} = \partial_B\tilde{\mathbf{F}}\otimes\mathbf{G}^B = \left(\partial_B\tilde{F}^{\alpha}_{\cdot A} + \Gamma^{\cdot\cdot\alpha}_{\beta\chi}\tilde{F}^{\chi}_{\cdot A}\tilde{F}^{\beta}_{\cdot B} - \overset{G}{\Gamma}{}^{\cdot\cdot C}_{BA}\tilde{F}^{\alpha}_{\cdot C}\right)\tilde{\mathbf{g}}_{\alpha}\otimes\mathbf{G}^{A}\otimes\mathbf{G}^{B} = \tilde{F}^{\alpha}_{\cdot A:B}\tilde{\mathbf{g}}_{\alpha}\otimes\mathbf{G}^{A}\otimes\mathbf{G}^{B}. \tag{219}$$

Note that unlike $\overset{G}{\nabla}\mathbf{F}$ in the first expression of (89), skew covariant components of $\overset{G}{\nabla}\tilde{\mathbf{F}}$ do not necessarily vanish:

$$\tilde{F}^{\alpha}_{\cdot[A:B]} = \partial_{[B}\tilde{F}^{\alpha}_{\cdot A]} + \Gamma^{\cdot\cdot\alpha}_{\beta\chi}\tilde{F}^{\beta}_{\cdot[B}\tilde{F}^{\chi}_{\cdot A]} - \overset{G}{\Gamma}{}^{\cdot\cdot C}_{[BA]}\tilde{F}^{\alpha}_{\cdot C} = \partial_{[B}\tilde{F}^{\alpha}_{\cdot A]} + \Gamma^{\cdot\cdot\alpha}_{[\beta\chi]}\tilde{F}^{\beta}_{\cdot B}\tilde{F}^{\chi}_{\cdot A}. \tag{220}$$

From (220), the material gradient $\overset{G}{\nabla}\tilde{\mathbf{F}}$ is generally symmetric in covariant indices only when both (108) holds and symmetric connection coefficients $\Gamma^{\cdot\cdot\alpha}_{\beta\chi} = \Gamma^{\cdot\cdot\alpha}_{(\beta\chi)}$ are prescribed on $\tilde{B}$. As an example of (218), letting $\mathbf{A}(x,t) = \bar{\mathbf{F}}^{-1}(x,t) = \bar{F}^{-1\alpha}_{\cdot a}\tilde{\mathbf{g}}_{\alpha}\otimes\mathbf{g}^{a}$, the spatial gradient is computed as

$$\overset{g}{\nabla}(\bar{\mathbf{F}}^{-1}) = \partial_b\bar{\mathbf{F}}^{-1}\otimes\mathbf{g}^b = \left(\partial_b\bar{F}^{-1\alpha}_{\cdot a} + \Gamma^{\cdot\cdot\alpha}_{\beta\chi}\bar{F}^{-1\chi}_{\cdot a}\bar{F}^{-1\beta}_{\cdot b} - \overset{g}{\Gamma}{}^{\cdot\cdot c}_{ba}\bar{F}^{-1\alpha}_{\cdot c}\right)\tilde{\mathbf{g}}_{\alpha}\otimes\mathbf{g}^{a}\otimes\mathbf{g}^{b} = \bar{F}^{-1\alpha}_{\cdot a:b}\tilde{\mathbf{g}}_{\alpha}\otimes\mathbf{g}^{a}\otimes\mathbf{g}^{b}. \tag{221}$$

Skew covariant components of $\overset{g}{\nabla}(\bar{\mathbf{F}}^{-1})$ do not necessarily vanish:

$$\bar{F}^{-1\alpha}_{\cdot[a:b]} = \partial_{[b}\bar{F}^{-1\alpha}_{\cdot a]} + \Gamma^{\cdot\cdot\alpha}_{\beta\chi}\bar{F}^{-1\beta}_{\cdot[b}\bar{F}^{-1\chi}_{\cdot a]} - \overset{g}{\Gamma}{}^{\cdot\cdot c}_{[ba]}\bar{F}^{-1\alpha}_{\cdot c} = \partial_{[b}\bar{F}^{-1\alpha}_{\cdot a]} + \Gamma^{\cdot\cdot\alpha}_{[\beta\chi]}\bar{F}^{-1\beta}_{\cdot b}\bar{F}^{-1\chi}_{\cdot a}. \tag{222}$$

From (222), the spatial gradient $\overset{g}{\nabla}(\bar{\mathbf{F}}^{-1})$ is generally symmetric in covariant indices only when both (112) holds and symmetric connection coefficients $\Gamma^{\cdot\cdot\alpha}_{\beta\chi} = \Gamma^{\cdot\cdot\alpha}_{(\beta\chi)}$ are prescribed on $\tilde{B}$.

Finally, consider a tensor field of order three or higher, with components referred to all three configurations $\tilde{B}$, B, and B_0, written $A^{\alpha\ldots\phi\ a\ldots f\ A\ldots F}_{\gamma\ldots\mu\ g\ldots m\ G\ldots M}$. By extension, the total covariant derivative of field $\mathbf{A}$ is

$$
\begin{aligned}
(A^{\alpha\ldots\phi\ a\ldots f\ A\ldots F}_{\gamma\ldots\mu\ g\ldots m\ G\ldots M})_{:\nu} &= \partial_\nu(A^{\alpha\ldots\phi\ a\ldots f\ A\ldots F}_{\gamma\ldots\mu\ g\ldots m\ G\ldots M})\\
&\quad + \Gamma^{\cdot\cdot\alpha}_{\nu\rho}A^{\rho\ldots\phi\ a\ldots f\ A\ldots F}_{\gamma\ldots\mu\ g\ldots m\ G\ldots M} + \cdots + \Gamma^{\cdot\cdot\phi}_{\nu\rho}A^{\alpha\ldots\rho\ a\ldots f\ A\ldots F}_{\gamma\ldots\mu\ g\ldots m\ G\ldots M}\\
&\quad - \Gamma^{\cdot\cdot\rho}_{\nu\gamma}A^{\alpha\ldots\phi\ a\ldots f\ A\ldots F}_{\rho\ldots\mu\ g\ldots m\ G\ldots M} - \cdots - \Gamma^{\cdot\cdot\rho}_{\nu\mu}A^{\alpha\ldots\phi\ a\ldots f\ A\ldots F}_{\gamma\ldots\rho\ g\ldots m\ G\ldots M}\\
&\quad + \bar{F}^{n}_{\cdot\nu}\overset{g}{\Gamma}{}^{\cdot\cdot a}_{nr}A^{\alpha\ldots\phi\ r\ldots f\ A\ldots F}_{\gamma\ldots\mu\ g\ldots m\ G\ldots M} + \cdots + \bar{F}^{n}_{\cdot\nu}\overset{g}{\Gamma}{}^{\cdot\cdot f}_{nr}A^{\alpha\ldots\phi\ A\ldots r\ A\ldots F}_{\gamma\ldots\mu\ g\ldots m\ G\ldots M}\\
&\quad - \bar{F}^{n}_{\cdot\nu}\overset{g}{\Gamma}{}^{\cdot\cdot r}_{ng}A^{\alpha\ldots\phi\ a\ldots f\ A\ldots F}_{\gamma\ldots\mu\ r\ldots m\ G\ldots M} - \cdots - \bar{F}^{n}_{\cdot\nu}\overset{g}{\Gamma}{}^{\cdot\cdot r}_{nm}A^{\alpha\ldots\phi\ a\ldots f\ A\ldots F}_{\gamma\ldots\mu\ g\ldots r\ G\ldots M}\\
&\quad + \tilde{F}^{-1N}_{\cdot\nu}\overset{G}{\Gamma}{}^{\cdot\cdot A}_{NR}A^{\alpha\ldots\phi\ a\ldots f\ R\ldots F}_{\gamma\ldots\mu\ g\ldots m\ G\ldots M} + \cdots + \tilde{F}^{-1N}_{\cdot\nu}\overset{G}{\Gamma}{}^{\cdot\cdot F}_{NR}A^{\alpha\ldots\phi\ a\ldots f\ A\ldots R}_{\gamma\ldots\mu\ g\ldots m\ G\ldots M}\\
&\quad - \tilde{F}^{-1N}_{\cdot\nu}\overset{G}{\Gamma}{}^{\cdot\cdot R}_{NG}A^{\alpha\ldots\phi\ a\ldots f\ A\ldots F}_{\gamma\ldots\mu\ g\ldots m\ R\ldots M} - \cdots - \tilde{F}^{-1N}_{\cdot\nu}\overset{G}{\Gamma}{}^{\cdot\cdot R}_{NM}A^{\alpha\ldots\phi\ a\ldots f\ A\ldots F}_{\gamma\ldots\mu\ g\ldots m\ G\ldots R}\\
&= (A^{\alpha\ldots\phi\ a\ldots f\ A\ldots F}_{\gamma\ldots\mu\ g\ldots m\ G\ldots M})_{:n}\bar{F}^{n}_{\cdot\nu}\\
&= (A^{\alpha\ldots\phi\ a\ldots f\ A\ldots F}_{\gamma\ldots\mu\ g\ldots m\ G\ldots M})_{:N}\tilde{F}^{-1N}_{\cdot\nu}.
\end{aligned}
\tag{223}
$$

Consider a differential line element $d\mathbf{X}$ in the reference configuration. Such an element is mapped to its representation in the intermediate configuration $d\tilde{\mathbf{x}}$ analogously to (86) via the Taylor series [14]

$$d\tilde{x}^{\alpha}(X) = (\tilde{F}^{\alpha}_{\cdot A})\Big|_X dX^A + \frac{1}{2!}(\tilde{F}^{\alpha}_{\cdot A:B})\Big|_X dX^A\,dX^B + \frac{1}{3!}(\tilde{F}^{\alpha}_{\cdot A:BC})\Big|_X dX^A\,dX^B\,dX^C + \cdots, \tag{224}$$

where components of the total covariant derivative of $\tilde{\mathbf{F}}$ are given in (219), and components of $\tilde{F}^{\alpha}_{\cdot A:BC}$ can be obtained through iteration of (217). Similarly, consider a differential line element $d\mathbf{x}$ in the current configuration. Such an element is mapped to its representation in the intermediate configuration $d\tilde{\mathbf{x}}$ via

$$d\tilde{x}^{\alpha}(x) = (\bar{F}^{-1\alpha}_{\cdot a})\Big|_{x} dx^{a} + \frac{1}{2!}(\bar{F}^{-1\alpha}_{\cdot a:b})\Big|_{x} dx^{a}\, dx^{b} + \frac{1}{3!}(\bar{F}^{-1\alpha}_{\cdot a:bc})\Big|_{x} dx^{a}\, dx^{b}\, dx^{c} + \cdots, \tag{225}$$

where components of the total covariant derivative of $\bar{\mathbf{F}}^{-1}$ are given in (221), and components of $\bar{F}^{-1\alpha}_{\cdot a:bc}$ can be obtained through iteration of (218). To first order in $d\mathbf{X}$ and $d\mathbf{x}$, and using (90) and (100), a standard assumption is

$$d\tilde{\mathbf{x}} = \tilde{\mathbf{F}}\, d\mathbf{X} = \bar{\mathbf{F}}^{-1}\mathbf{F}\, d\mathbf{X} = \bar{\mathbf{F}}^{-1}\, d\mathbf{x}. \tag{226}$$

3.2.4. Divergence, curl, and Laplacian. Definitions that follow correspond to anholonomic space $\tilde{B}$, for which (179)–(183) apply for covariant and partial differentiation. The divergence of a contravariant vector field $\mathbf{V} = V^{\alpha}\tilde{\mathbf{g}}_{\alpha}$ is

$$\langle \nabla, \mathbf{V} \rangle = \mathrm{tr}\,(\nabla \mathbf{V}) = \langle \partial_{\alpha}\mathbf{V}, \tilde{\mathbf{g}}^{\alpha} \rangle = \nabla_{\alpha} V^{\alpha} = \partial_{\alpha} V^{\alpha} + \Gamma^{\cdot\cdot\alpha}_{\alpha\beta} V^{\beta}. \tag{227}$$

Note that the analog of the final equality in (66) does not necessarily apply here, since (52) may not hold for connection coefficients on $\tilde{B}$ (i.e. $\Gamma^{\cdot\cdot\alpha}_{\beta\chi}$ are not necessarily Levi-Civita connection coefficients). The vector cross product $\times$ obeys, for two vectors $\mathbf{V}$ and $\mathbf{W}$ and two covectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$,

$$\mathbf{V} \times \mathbf{W} = \epsilon_{\alpha\beta\chi} V^{\beta} W^{\chi} \tilde{\mathbf{g}}^{\alpha}, \qquad \boldsymbol{\alpha} \times \boldsymbol{\beta} = \epsilon^{\alpha\beta\chi} \alpha_{\beta} \beta_{\chi} \tilde{\mathbf{g}}_{\alpha}. \tag{228}$$

The curl of a covariant vector field $\boldsymbol{\alpha} = \alpha_{\alpha}\tilde{\mathbf{g}}^{\alpha}$ is then defined as

$$\nabla \times \boldsymbol{\alpha} = \tilde{\mathbf{g}}^{\alpha} \times \partial_{\alpha}(\alpha_{\beta}\tilde{\mathbf{g}}^{\beta}) = \tilde{\mathbf{g}}^{\alpha} \times \tilde{\mathbf{g}}^{\beta} \nabla_{\alpha}\alpha_{\beta} = \epsilon^{\alpha\beta\chi} \nabla_{\alpha}\alpha_{\beta} \tilde{\mathbf{g}}_{\chi} = \epsilon^{\alpha\beta\chi} \nabla_{\beta}\alpha_{\chi} \tilde{\mathbf{g}}_{\alpha}. \tag{229}$$

A relationship like the final equality in (68) does not necessarily hold here, since coefficients $\Gamma^{\cdot\cdot\alpha}_{\beta\chi}$ are possibly non-symmetric. The Laplacian of a scalar field f is, from the symmetry of inverse metric $\tilde{g}^{\alpha\beta}$,

$$\nabla^{2} f = \tilde{g}^{\alpha\beta} \nabla_{\alpha}\nabla_{\beta} f = \tilde{g}^{\alpha\beta} \nabla_{\beta}\nabla_{\alpha} f = \tilde{g}^{\alpha\beta} \nabla_{\beta}(\partial_{\alpha} f) = \tilde{g}^{\alpha\beta} \nabla_{\alpha}(\partial_{\beta} f). \tag{230}$$

An identity such as the third equality in (69) does not necessarily apply here. Unlike (70), the divergence of the curl of a (co)vector field does not necessarily vanish in an anholonomic space:

$$\langle \nabla, \nabla \times \boldsymbol{\alpha} \rangle = \nabla_{\alpha}(\epsilon^{\alpha\beta\chi} \nabla_{\beta}\alpha_{\chi}) = \nabla_{\alpha}[(1/\sqrt{\tilde{g}})] e^{\alpha\beta\chi} \nabla_{\beta}\alpha_{\chi} - \frac{1}{2\sqrt{\tilde{g}}} e^{\alpha\beta\chi} [R^{\cdot\cdot\cdot\delta}_{\alpha\beta\chi}\alpha_{\delta} + 2T^{\cdot\cdot\delta}_{\alpha\beta}\nabla_{\delta}\alpha_{\chi}]. \tag{231}$$

Identity (189) has been used. Similarly, from (188), the curl of the gradient of a scalar field may be non-zero in anholonomic coordinates:

$$\nabla \times \nabla f = \epsilon^{\alpha\beta\chi}(\nabla_{\beta}\nabla_{\chi} f)\tilde{\mathbf{g}}_{\alpha} = \epsilon^{\alpha\beta\chi}(T^{\cdot\cdot\gamma}_{\chi\beta}\partial_{\gamma} f)\tilde{\mathbf{g}}_{\alpha}. \tag{232}$$

Identities (231) and (232) are now examined for the four particular sets of connection coefficients $\Gamma^{\cdot\cdot\alpha}_{\beta\chi}$ defined in Section 3.2.2.

When configuration $\tilde{B}$ is a Euclidean space and (190) defines the connection coefficients, then the right-hand sides of (231) and (232) are identically zero.

When (196) defines the connection coefficients, i.e. when identical (possibly curvilinear) coordinate systems are used in reference and intermediate configurations, then (231) and (232) become

$$\begin{aligned}\langle \nabla, \nabla \times \boldsymbol{\alpha} \rangle = \frac{1}{\sqrt{G}} e^{\alpha\beta\chi} [&\delta^{C}_{\chi}\delta^{\delta}_{D}\tilde{F}^{-1A}_{\cdot[\alpha}\tilde{F}^{-1B}_{\cdot\beta]}\partial_{A}({}^{G}\Gamma^{\cdot\cdot D}_{BC})\alpha_{\delta} - (\delta^{\delta}_{D}\,{}^{G}\Gamma^{\cdot\cdot D}_{AB}\tilde{F}^{-1A}_{\cdot[\alpha}\delta^{B}_{\beta]} + \tilde{F}^{-1A}_{\cdot[\alpha}\tilde{F}^{-1B}_{\cdot\beta]}\partial_{A}\tilde{F}^{\delta}_{\cdot B}) \\ &\times \tilde{F}^{-1D}_{\cdot\delta}(\partial_{D}\alpha_{\chi} - \delta^{\epsilon}_{E}\delta^{C}_{\chi}\,{}^{G}\Gamma^{\cdot\cdot E}_{DC}\alpha_{\epsilon})],\end{aligned} \tag{233}$$

$$\nabla \times \nabla f = -\frac{1}{\sqrt{G}} e^{\alpha\beta\chi} [(\delta^{\gamma}_{A}\,{}^{G}\Gamma^{\cdot\cdot A}_{BC}\tilde{F}^{-1B}_{\cdot[\beta}\delta^{C}_{\chi]} + \tilde{F}^{-1A}_{\cdot[\beta}\tilde{F}^{-1B}_{\cdot\chi]}\partial_{A}\tilde{F}^{\gamma}_{\cdot B})\tilde{F}^{-1E}_{\cdot\gamma}\partial_{E} f]\delta^{F}_{\alpha}\mathbf{G}_{F}. \tag{234}$$

Definitions (198) and (199) have been used for torsion and curvature. Recall that neither need be zero even if $\tilde{\mathbf{F}}$ is integrable. Vanishing of the covariant derivative of $\tilde{g} = \det(\tilde{g}_{\alpha\beta})$ follows from (200).

Analogously, when (206) defines the connection coefficients, i.e. when identical (possibly curvilinear) coordinate systems are used in current and intermediate configurations, then (231) and (232) become

$$\begin{aligned}\langle \nabla, \nabla \times \boldsymbol{\alpha} \rangle = \frac{1}{\sqrt{g}} e^{\alpha\beta\chi} [\delta^c_\chi \delta^\delta_d \bar{F}^a_{\cdot[\alpha} \bar{F}^b_{\cdot\beta]} \partial_a ({}^{g}\Gamma^{\cdot\cdot d}_{bc}) \alpha_\delta - (\delta^\delta_d {}^{g}\Gamma^{\cdot\cdot d}_{ab} \bar{F}^a_{\cdot[\alpha} \delta^b_{\beta]} + \bar{F}^a_{\cdot[\alpha} \bar{F}^b_{\cdot\beta]} \partial_a \bar{F}^{-1\delta}{}_{\cdot b}) \\ \times \bar{F}^d_{\cdot\delta} (\partial_d \alpha_\chi - \delta^\epsilon_e \delta^c_\chi {}^{g}\Gamma^{\cdot\cdot e}_{dc} \alpha_\epsilon)],\end{aligned} \tag{235}$$

$$\nabla \times \nabla f = -\frac{1}{\sqrt{g}} e^{\alpha\beta\chi} [(\delta^\gamma_a {}^{g}\Gamma^{\cdot\cdot a}_{bc} \bar{F}^b_{\cdot[\beta} \delta^c_{\chi]} + \bar{F}^a_{\cdot[\beta} \bar{F}^b_{\cdot\chi]} \partial_a \bar{F}^{-1\gamma}{}_{\cdot b}) \bar{F}^e_{\cdot\gamma} \partial_e f] \delta^f_\alpha \mathbf{g}_f. \tag{236}$$

Definitions (208) and (209) have been used for torsion and curvature. Recall that neither need be zero even if $\bar{\mathbf{F}}^{-1}$ is integrable. Vanishing of the covariant derivative of $\tilde{g} = \det(\tilde{g}_{\alpha\beta})$ follows from (210).

Finally, consider the case when (214) defines the connection coefficients, i.e. when a Cartesian system is used in the intermediate configuration, such that $\Gamma^{\cdot\cdot\alpha}_{\beta\chi} = 0$ identically. In that case, (231) and (232) become, respectively,

$$\langle \nabla, \nabla \times \boldsymbol{\alpha} \rangle = \nabla_\alpha (\epsilon^{\alpha\beta\chi} \nabla_\beta \alpha_\chi) = e^{\alpha\beta\chi} \nabla_\alpha (\nabla_\beta \alpha_\chi) = e^{\alpha\beta\chi} \partial_{[\alpha} (\partial_{\beta]} \alpha_\chi) = e^{\alpha\beta\chi} \kappa^{\cdot\cdot\delta}_{\beta\alpha} \partial_\delta \alpha_\chi, \tag{237}$$

$$\nabla \times \nabla f = \epsilon^{\alpha\beta\chi} (\nabla_\beta \nabla_\chi f) \tilde{\mathbf{g}}_\alpha = e^{\alpha\beta\chi} [\partial_{[\beta} (\partial_{\chi]} f)] \mathbf{e}_\alpha = e^{\alpha\beta\chi} [\kappa^{\cdot\cdot\delta}_{\chi\beta} \partial_\delta f] \mathbf{e}_\alpha. \tag{238}$$

Thus, even if intermediate connection coefficients vanish, the divergence of the curl and the curl of the gradient need not be identically zero in anholonomic coordinates.

3.3. Anholonomic Jacobian determinants

3.3.1. Jacobian determinant of $\tilde{\mathbf{F}}$. Jacobian determinant $\tilde{J}$ provides the relationship between differential volume elements in reference and intermediate configurations. Differential volume element in the intermediate configuration ($d\tilde{V}$) is

$$\tilde{J}\, dV = d\tilde{V} = \sqrt{\tilde{g}}\, d\tilde{x}^1\, d\tilde{x}^2\, d\tilde{x}^3 \subset \tilde{B}. \tag{239}$$

The Jacobian determinant $\tilde{J}[\tilde{\mathbf{F}}(X,t), \tilde{g}, G(X)]$ is [14]

$$\tilde{J} = \frac{1}{6} \epsilon^{ABC} \epsilon_{\alpha\beta\chi}\, \tilde{F}^\alpha_{\cdot A} \tilde{F}^\beta_{\cdot B} \tilde{F}^\chi_{\cdot C} = \frac{1}{6} \sqrt{\tilde{g}/G}\, e^{ABC} e_{\alpha\beta\chi} \tilde{F}^\alpha_{\cdot A} \tilde{F}^\beta_{\cdot B} \tilde{F}^\chi_{\cdot C} = \sqrt{\tilde{g}/G}\ \det \tilde{\mathbf{F}} = \sqrt{\frac{\det(\tilde{g}_{\alpha\beta})}{\det(G_{AB})}}\ \det(\tilde{F}^\alpha_{\cdot A}). \tag{240}$$

From similar arguments, inverse Jacobian determinant $\tilde{J}^{-1}[\tilde{\mathbf{F}}^{-1}(x,t), G(X), \tilde{g}] = 1/\tilde{J}$ is

$$\tilde{J}^{-1} = \frac{1}{6} \epsilon^{\alpha\beta\chi} \epsilon_{ABC}\, \tilde{F}^{-1A}{}_{\cdot\alpha} \tilde{F}^{-1B}{}_{\cdot\beta} \tilde{F}^{-1C}{}_{\cdot\chi} = \sqrt{G/\tilde{g}}\ \det \tilde{\mathbf{F}}^{-1} = \sqrt{\det(G_{AB})/\det(\tilde{g}_{\alpha\beta})}\ \det(\tilde{F}^{-1A}{}_{\cdot\alpha}). \tag{241}$$

When the mapping is restricted to rigid translation (or to no motion at all), then $\tilde{F}^\alpha_{\cdot A} = g^\alpha_A$ [the shifter of (129)], $\tilde{\mathbf{F}} = g^\alpha_A \tilde{\mathbf{g}}_\alpha \otimes \mathbf{G}^A$, and $\tilde{J} = \sqrt{\tilde{g}/G}\det(g^\alpha_A) = 1$ follows from (133). From identity (51), it follows that

$$\frac{\partial \tilde{J}}{\partial \tilde{F}^\alpha_{\cdot A}} = \tilde{J} \tilde{F}^{-1A}{}_{\cdot\alpha}, \qquad \frac{\partial \tilde{J}^{-1}}{\partial \tilde{F}^{-1A}{}_{\cdot\alpha}} = \tilde{J}^{-1} \tilde{F}^\alpha_{\cdot A}. \tag{242}$$

Taking the divergence of the first of (242) results in [14]

$$(\tilde{J} \tilde{F}^{-1A}{}_{\cdot\alpha})_{:A} = \frac{1}{2} \epsilon_{\alpha\beta\chi} \epsilon^{ABC} \left(\tilde{F}^\beta_{\cdot B} \tilde{F}^\chi_{\cdot C:A} + \tilde{F}^\beta_{\cdot B:A} \tilde{F}^\chi_{\cdot C} \right) = \epsilon_{\alpha\beta\chi} \epsilon^{ABC} \tilde{F}^\beta_{\cdot A} \tilde{F}^\chi_{\cdot [B:C]}. \tag{243}$$

Recall from (220) that material gradient $\overset{G}{\nabla}\tilde{\mathbf{F}}$ is generally symmetric in covariant indices only when both (108) holds and symmetric coefficients $\Gamma^{\cdot\cdot\alpha}_{\beta\chi} = \Gamma^{\cdot\cdot\alpha}_{(\beta\chi)}$ are prescribed on $\tilde{B}$. Thus, unless such symmetry conditions hold, the analog of Piola's identity (97) does not necessarily apply for the divergence of $\tilde{J}\tilde{F}^{-1A}_{\cdot\alpha}$.

Let the vector field $\mathbf{A} = A^A\mathbf{G}_A$ be the Piola transform of $\tilde{\mathbf{a}} = \tilde{a}^\alpha\tilde{\mathbf{g}}_\alpha$:

$$A^A = \tilde{J}\tilde{F}^{-1A}_{\cdot\alpha}\tilde{a}^\alpha. \tag{244}$$

Taking the divergence of (244) and applying the product rule for covariant differentiation with definition (217) results in

$$A^A_{;A} = A^A_{:A} = (\tilde{J}\tilde{F}^{-1A}_{\cdot\alpha})_{:A}\tilde{a}^\alpha + \tilde{J}\tilde{F}^{-1A}_{\cdot\alpha}\tilde{a}^\alpha_{:A} = \tilde{J}[\tilde{J}^{-1}(\tilde{J}\tilde{F}^{-1A}_{\cdot\alpha})_{:A}\tilde{a}^\alpha + \tilde{a}^\alpha_{:\alpha}] = \tilde{J}\,\tilde{\nabla}_\alpha\tilde{a}^\alpha, \tag{245}$$

where the anholonomic covariant derivative is defined as [14, 19]

$$\tilde{\nabla}_\alpha(\cdot) = (\cdot)_{:A}\tilde{F}^{-1A}_{\cdot\alpha} + (\cdot)\tilde{J}^{-1}(\tilde{J}\tilde{F}^{-1A}_{\cdot\alpha})_{:A} = (\cdot)_{:\alpha} + (\cdot)\tilde{J}^{-1}\epsilon_{\alpha\beta\chi}\epsilon^{ABC}\tilde{F}^\beta_{\cdot A}\tilde{F}^\chi_{\cdot B:C}. \tag{246}$$

The second term on the right-hand side of (246) vanishes when the right-hand side of (243) vanishes, in which case (245) becomes similar to (99).

As a second example, consider the following Piola transformation between contravariant second-order tensor $\mathbf{A} = A^{AB}\mathbf{G}_A \otimes \mathbf{G}_B$ and $\tilde{\mathbf{a}} = \tilde{a}^{A\alpha}\mathbf{G}_A \otimes \tilde{\mathbf{g}}_\alpha$:

$$A^{AB} = \tilde{J}\tilde{F}^{-1B}_{\cdot\alpha}\tilde{a}^{A\alpha}. \tag{247}$$

Divergence on the second leg of $\mathbf{A}$ is computed as

$$\begin{aligned} A^{AB}_{;B} &= A^{AB}_{:B} = (\tilde{J}\tilde{F}^{-1B}_{\cdot\alpha})_{:B}\tilde{a}^{A\alpha} + \tilde{J}\tilde{F}^{-1B}_{\cdot\alpha}\tilde{a}^{A\alpha}_{:B} = \tilde{J}\,\tilde{\nabla}_\alpha\tilde{a}^{A\alpha} \\ &= \tilde{J}(\partial_\alpha\tilde{a}^{A\alpha} + \Gamma^{\cdot\cdot\alpha}_{\alpha\beta}\tilde{a}^{A\beta} + \overset{G}{\Gamma}{}^{\cdot\cdot A}_{BC}\tilde{F}^{-1B}_{\cdot\alpha}\tilde{a}^{C\alpha}) + \epsilon_{\alpha\beta\chi}\epsilon^{DBC}\tilde{a}^{A\alpha}\tilde{F}^\beta_{\cdot D}(\partial_C\tilde{F}^\chi_{\cdot B} + \Gamma^{\cdot\cdot\chi}_{\delta\epsilon}\tilde{F}^\delta_{\cdot C}\tilde{F}^\epsilon_{\cdot B}). \end{aligned} \tag{248}$$

Physically, the quantity $\tilde{J}$ is associated with volume changes resulting from mechanisms associated with mapping $\tilde{\mathbf{F}}$. When $\tilde{\mathbf{F}}$ represents plastic deformation from dislocation glide [3, 43], or shear associated with twinning [19, 20], then $\tilde{\mathbf{F}}$ is isochoric and $\tilde{J} = 1$. When $\tilde{\mathbf{F}}$ represents thermal deformation [15], and when such thermal deformation is isotropic (e.g. for cubic crystals), then $\tilde{\mathbf{F}} = \tilde{J}^{1/3}\mathbf{1} = \tilde{J}^{1/3}g^\alpha_{\cdot A}\tilde{\mathbf{g}}_\alpha \otimes \mathbf{G}^A$, where $\tilde{J}$ depends on temperature [25]. Mapping $\tilde{\mathbf{F}}$ exhibits a similar spherical form when it represents isotropic deformation from voids [18] or point defects [17], in which case $\tilde{J}$ depends on the volume fraction of such defects. When $\tilde{\mathbf{F}}$ depicts growth of biomaterials, then $\tilde{J}$ reflects both density changes and mass changes, with the former often negligible in the context of tissue mechanics [25].

3.3.2. Jacobian determinant of $\bar{\mathbf{F}}$. Definitions and arguments of the previous section can be repeated for the Jacobian determinant of $\bar{\mathbf{F}}$. Differential volume elements in spatial and intermediate configurations are related by

$$dv = \bar{J}\,d\tilde{V}, \tag{249}$$

where the Jacobian determinant $\bar{J}[\bar{\mathbf{F}}(X,t), g(x), \tilde{g}]$ is

$$\bar{J} = \frac{1}{6}\epsilon_{abc}\epsilon^{\alpha\beta\chi}\;\bar{F}^a_{\cdot\alpha}\bar{F}^b_{\cdot\beta}\bar{F}^c_{\cdot\chi} = \sqrt{g/\tilde{g}}\;\det\bar{\mathbf{F}} = \sqrt{\det(g_{ab})/\det(\tilde{g}_{\alpha\beta})}\;\det(\bar{F}^a_{\cdot\alpha}). \tag{250}$$

From similar arguments, inverse Jacobian determinant $\bar{J}^{-1}[\bar{\mathbf{F}}^{-1}(x,t), \tilde{g}, g(x)] = 1/\bar{J}$ is

$$\bar{J}^{-1} = \frac{1}{6}\epsilon_{\alpha\beta\chi}\epsilon^{abc}\;\bar{F}^{-1\alpha}_{\cdot a}\bar{F}^{-1\beta}_{\cdot b}\bar{F}^{-1\chi}_{\cdot c} = \sqrt{\tilde{g}/g}\;\det\bar{\mathbf{F}}^{-1} = \sqrt{\det(\tilde{g}_{\alpha\beta})/\det(g_{ab})}\;\det(\bar{F}^{-1\alpha}_{\cdot a}). \tag{251}$$

Note that products of Jacobian determinants and their inverses obey

$$J = \bar{J}\,\tilde{J}, \quad J^{-1} = \tilde{J}^{-1}\bar{J}^{-1}. \tag{252}$$

From identity (51), it follows that, analogously to (242),

$$\frac{\partial \bar{J}}{\partial \bar{F}^{a}_{.\alpha}} = \bar{J}\bar{F}^{-1\alpha}_{\quad .a}, \quad \frac{\partial \bar{J}^{-1}}{\partial \bar{F}^{-1\alpha}_{\quad .a}} = \bar{J}^{-1}\bar{F}^{a}_{.\alpha}. \tag{253}$$

Taking the divergence of the first of (253) results in

$$(\bar{J}^{-1}\bar{F}^{a}_{.\alpha})_{:a} = \frac{1}{2}\epsilon_{\alpha\beta\chi}\epsilon^{abc}\left(\bar{F}^{-1\beta}_{\quad .b}\bar{F}^{-1\chi}_{\quad .c:a} + \bar{F}^{-1\beta}_{\quad .b:a}\bar{F}^{-1\chi}_{\quad .c}\right) = \epsilon_{\alpha\beta\chi}\epsilon^{abc}\bar{F}^{-1\beta}_{\quad .a}\bar{F}^{-1\chi}_{\quad .[b:c]}. \tag{254}$$

Recall from (222) that spatial gradient $\overset{g}{\nabla}(\bar{\mathbf{F}}^{-1})$ is generally symmetric in covariant indices only when both (112) holds and symmetric connection coefficients $\Gamma^{..\alpha}_{\beta\chi} = \Gamma^{..\alpha}_{(\beta\chi)}$ are prescribed on $\tilde{B}$. Only in such cases does the divergence of $\bar{J}^{-1}\bar{F}^{a}_{.\alpha}$ generally vanish, such that an identity akin to (96) holds.

Let the vector field $\mathbf{A} = A^{a}\mathbf{g}_{a}$ be the Piola transform of $\tilde{\mathbf{a}} = \tilde{a}^{\alpha}\tilde{\mathbf{g}}_{\alpha}$:

$$A^{a} = \bar{J}^{-1}\bar{F}^{a}_{.\alpha}\tilde{a}^{\alpha}. \tag{255}$$

Taking the divergence of (255) and applying the product rule for covariant differentiation with (218),

$$A^{a}_{;a} = A^{a}_{:a} = (\bar{J}^{-1}\bar{F}^{a}_{.\alpha})_{:a}\tilde{a}^{\alpha} + \bar{J}^{-1}\bar{F}^{a}_{.\alpha}\tilde{a}^{\alpha}_{:a} = \bar{J}^{-1}[\bar{J}(\bar{J}^{-1}\bar{F}^{a}_{.\alpha})_{:a}\tilde{a}^{\alpha} + \tilde{a}^{\alpha}_{:\alpha}] = \bar{J}^{-1}\,\bar{\nabla}_{\alpha}\tilde{a}^{\alpha}, \tag{256}$$

where the anholonomic covariant derivative is defined analogously to (246):

$$\bar{\nabla}_{\alpha}(\cdot) = (\cdot)_{:a}\bar{F}^{a}_{.\alpha} + (\cdot)\bar{J}(\bar{J}^{-1}\bar{F}^{a}_{.\alpha})_{:a} = (\cdot)_{:\alpha} + (\cdot)\bar{J}\epsilon_{\alpha\beta\chi}\epsilon^{abc}\bar{F}^{-1\beta}_{\quad .a}\bar{F}^{-1\chi}_{\quad .b:c}. \tag{257}$$

The second term on the right-hand side of (257) vanishes when the right-hand side of (254) vanishes, in which case (256) becomes similar to (99).

As a second example, consider the following Piola transformation between contravariant second-order tensor $\mathbf{A} = A^{ab}\mathbf{g}_{a}\otimes\mathbf{g}_{b}$ and $\tilde{\mathbf{a}} = \tilde{a}^{a\alpha}\mathbf{g}_{a}\otimes\tilde{\mathbf{g}}_{\alpha}$:

$$A^{ab} = \bar{J}^{-1}\bar{F}^{b}_{.\alpha}\tilde{a}^{a\alpha}. \tag{258}$$

Divergence on the second leg of $\mathbf{A}$ is computed similarly to (248):

$$\begin{aligned} A^{ab}_{\ \ ;b} = A^{ab}_{\ \ :b} &= (\bar{J}^{-1}\bar{F}^{b}_{.\alpha})_{:b}\tilde{a}^{a\alpha} + \bar{J}^{-1}\bar{F}^{b}_{.\alpha}\tilde{a}^{a\alpha}_{\ \ :b} = \bar{J}^{-1}\,\bar{\nabla}_{\alpha}\tilde{a}^{a\alpha} \\ &= \bar{J}^{-1}(\partial_{\alpha}\tilde{a}^{a\alpha} + \Gamma^{..\alpha}_{\alpha\beta}\tilde{a}^{a\beta} + \overset{g}{\Gamma}{}^{..a}_{bc}\bar{F}^{b}_{.\alpha}\tilde{a}^{c\alpha}) + \epsilon_{\alpha\beta\chi}\epsilon^{dbc}\tilde{a}^{a\alpha}\bar{F}^{-1\beta}_{\quad .d}(\partial_{c}\bar{F}^{-1\chi}_{\quad .b} + \Gamma^{..\chi}_{\delta\epsilon}\bar{F}^{-1\delta}_{\quad .c}\bar{F}^{-1\epsilon}_{\quad .b}). \end{aligned} \tag{259}$$

From (79) and (96), it follows that covariant derivative operations in (246) and (257) are equivalent:

$$\bar{\nabla}_{\alpha}(\cdot) = (\cdot)_{:a}\bar{F}^{a}_{.\alpha} + (\cdot)\bar{J}(\bar{J}^{-1}\bar{F}^{a}_{.\alpha})_{:a} = (\cdot)_{:A}\bar{F}^{a}_{.\alpha}F^{-1A}_{\quad .a} + (\cdot)\bar{J}(J^{-1}\tilde{J}F^{a}_{.A}\tilde{F}^{-1A}_{\quad .\alpha})_{:a} = \tilde{\nabla}_{\alpha}(\cdot). \tag{260}$$

When $\bar{\mathbf{F}}$ physically represents elastic deformation, then $\bar{J}$ quantifies elastic volume changes typically related to pressure changes. The physical importance of identities associated with Piola transformations such as those above will become clear in the context of the balance of linear momentum discussed in Section 4. Derivations of Piola transformations and related identities in anholonomic configurations have been given elsewhere in coordinate-free notation [6, 40].

3.4. Dislocation theory

Developments of Sections 3.1–3.3 have known and immediate relevance in the context of continuum dislocation theory. Connection coefficients referred to referential and spatial coordinate systems are introduced as

$$\tilde{\Gamma}^{..A}_{BC} = \tilde{F}^{-1A}_{\quad .\alpha}\partial_{B}\tilde{F}^{\alpha}_{.C} = -\tilde{F}^{\alpha}_{.C}\tilde{F}^{\beta}_{.B}\partial_{\beta}\tilde{F}^{-1A}_{\quad .\alpha}, \quad \bar{\Gamma}^{..a}_{bc} = \bar{F}^{a}_{.\alpha}\partial_{b}\bar{F}^{-1\alpha}_{\quad .c} = -\bar{F}^{-1\alpha}_{\quad .c}\bar{F}^{-1\beta}_{\quad .B}\partial_{\beta}\bar{F}^{a}_{.\alpha}. \tag{261}$$

Torsion tensors of these coefficients are related to the equivalent anholonomic object of (176) and (184) as follows:

$$\tilde{T}^{..A}_{BC} = \tilde{\Gamma}^{..A}_{[BC]} = \tilde{F}^{-1A}_{\quad .\alpha}\tilde{F}^{\beta}_{.B}\tilde{F}^{\chi}_{.C}\tilde{\kappa}^{..\alpha}_{\beta\chi}, \qquad \bar{T}^{..a}_{bc} = \bar{\Gamma}^{..a}_{[bc]} = \bar{F}^{a}_{.\alpha}\bar{F}^{-1\beta}_{\quad .b}\bar{F}^{-1\chi}_{\quad .c}\bar{\kappa}^{..\alpha}_{\beta\chi}. \tag{262}$$

Riemann–Christoffel curvature tensors formed from connections in (261), which are said to be integrable, vanish identically at all points $X \in B_0$ or $x \in B$ (see [27]). Let C be a closed curve enclosing simply connected area S in configuration B_0, and let $\tilde{c}$ be the image of this curve in $\tilde{B}$. Assigning constant basis vectors $\tilde{\mathbf{g}}_\alpha$ over C and S, a measure of incompatibility associated with $\tilde{\mathbf{F}}$ in domain S is obtained using (226) and Stokes's theorem:

$$\oint_{\tilde{c}} d\tilde{x}^\alpha = \oint_C \tilde{F}^\alpha_{\cdot A}\, dX^A = \int_S \epsilon^{ABC} \tilde{F}^\alpha_{\cdot A;C} N_B\, dS = \int_S \epsilon^{ABC} \partial_A \tilde{F}^\alpha_{\cdot B} N_C\, dS = \int_S \tilde{\alpha}^{\alpha C} N_C\, dS, \tag{263}$$

where $N_C(X)$ are components of the unit normal to S. Analogously, letting c be a closed curve enclosing simply connected area s in the current configuration,

$$\oint_{\tilde{c}} d\tilde{x}^\alpha = \oint_c \bar{F}^{-1\alpha}_{\cdot a}\, dx^a = \int_s \epsilon^{abc} \bar{F}^{-1\alpha}_{\cdot a;c} n_b\, ds = \int_s \epsilon^{abc} \partial_a \bar{F}^{-1\alpha}_{\cdot b} n_c\, ds = \int_s \bar{\alpha}^{\alpha c} n_c\, ds, \tag{264}$$

with $n_c(x)$ the unit normal to s. Let $\bar{\mathbf{F}}$ represent elastic deformation, and let $\tilde{\mathbf{F}}$ represent plastic deformation from dislocation motion. Dislocation density tensors $\tilde{\alpha}^{\alpha A}$ and $\bar{\alpha}^{\alpha a}$ are related to torsion tensors of (262) as

$$\tilde{\alpha}^{\alpha C} = \epsilon^{ABC} \partial_A \tilde{F}^\alpha_{\cdot B} = \epsilon^{ABC} \tilde{F}^\alpha_{\cdot D} \tilde{T}^{\cdot\cdot D}_{BC}, \qquad \bar{\alpha}^{\alpha c} = \epsilon^{abc} \partial_a \bar{F}^{-1\alpha}_{\cdot b} = \epsilon^{abc} \bar{F}^{-1\alpha}_{\cdot d} \bar{T}^{\cdot\cdot d}_{bc}. \tag{265}$$

It follows from identities $\tilde{\kappa}^{\cdot\cdot\alpha}_{\beta\chi} = \bar{\kappa}^{\cdot\cdot\alpha}_{\beta\chi}$ of (176), (252), (262), and $J\epsilon^{abc} = \epsilon^{ABC} F^a_{\cdot A} F^b_{\cdot B} F^c_{\cdot C}$ that dislocation density tensors are related by

$$\tilde{J}^{-1} \tilde{F}^\alpha_{\cdot A} \tilde{\alpha}^{\beta A} = \bar{J} \bar{F}^{-1\alpha}_{\cdot a} \bar{\alpha}^{\beta a}. \tag{266}$$

Integrals (263) and (264) can be interpreted as total Burgers vectors of all dislocation lines piercing areas S and s, respectively [3]. Many other dislocation density tensors can be derived by mapping those in (265) to various configurations [6, 32, 40, 42]; sign conventions can also vary among definitions of the total Burgers vector and dislocation density tensor. A relationship between torsion and dislocations has also been described in the context of gauge theory [44]. In constitutive theories of crystalline solids of the gradient type, geometrically necessary dislocation density tensors such as those listed here may directly affect stored energy [42, 43, 45] and strain hardening [9, 42, 43, 45].

4. Balance of linear momentum

For a hyperelastic–plastic material with uniform properties, it is shown that the static local balance of linear momentum, when mapped to the intermediate configuration, can be written directly in terms of $\bar{\mathbf{F}}$ and its covariant derivatives. Acceleration and body forces are excluded; these can be incorporated into subsequent developments without difficulty.

4.1. Local momentum balance

Let $\boldsymbol{\sigma}(x,t) = \sigma^{ab} \mathbf{g}_a \otimes \mathbf{g}_b$ denote the symmetric Cauchy stress referred to configuration B. In the absence of body forces and acceleration, the local balance of linear momentum is, in possibly curvilinear spatial coordinates x^a, [37–39]

$$\sigma^{ab}_{;b} = 0, \tag{267}$$

recalling that the subscripted semicolon denotes covariant differentiation as in (55). Defining the Piola transform with respect $\bar{\mathbf{F}}$ to similarly to (258),

$$\sigma^{ab} = \bar{J}^{-1} \bar{F}^b_{\cdot\alpha} P^{a\alpha}, \quad P^{a\alpha} = \bar{J} \bar{F}^{-1\alpha}_{\cdot b} \sigma^{ab}, \tag{268}$$

and appealing to (259), momentum balance (267) can be written, upon multiplication by $\bar{J}$, as

$$\begin{aligned} 0 &= \bar{J} \sigma^{ab}_{;b} = \bar{\nabla}_\alpha P^{a\alpha} \\ &= \partial_\alpha P^{a\alpha} + \Gamma^{\cdot\cdot\alpha}_{\alpha\beta} P^{a\beta} + \overset{g}{\Gamma}{}^{\cdot\cdot a}_{bc} \bar{F}^b_{\cdot\alpha} P^{c\alpha} + \bar{J} \epsilon_{\alpha\beta\chi} \epsilon^{dbc} P^{a\alpha} \bar{F}^{-1\beta}_{\cdot d} (\partial_c \bar{F}^{-1\chi}_{\cdot b} + \Gamma^{\cdot\cdot\chi}_{\delta\epsilon} \bar{F}^{-1\delta}_{\cdot c} \bar{F}^{-1\epsilon}_{\cdot b}). \end{aligned} \tag{269}$$

The first term in parentheses on the right-hand side of (269) is related to a dislocation density in (265) via $\epsilon^{bcd} \partial_b \bar{F}^{-1\chi}_{\cdot c} = \bar{\alpha}^{\chi d}$. While the balance of linear momentum in anholonomic space has been considered elsewhere [6, 7, 23, 40], several particular forms derived in curvilinear coordinates in what follows are believed to be new.

4.2. Hyperelastic–plastic material

Consider a crystalline solid. Let $\bar{\mathbf{F}}$ physically represent elastic lattice deformation, and let $\tilde{\mathbf{F}}$ physically represent plastic deformation, e.g. resulting from glide of crystal dislocations. Let Ψ denote the strain energy per unit volume of material in intermediate configuration $\tilde{B}$, which is assumed to be a function of the following arguments:

$$\Psi = \Psi(\bar{\mathbf{C}}, \tilde{\mathbf{g}}) = \Psi[\bar{\mathbf{C}}(\bar{\mathbf{F}}, \mathbf{g}), \tilde{\mathbf{g}}], \tag{270}$$

where $\mathbf{g}$ denotes the spatial metric tensor of (11) with components $g_{ab}(x)$, $\tilde{\mathbf{g}}$ denotes the intermediate metric tensor of (122) with components $\tilde{g}_{\alpha\beta}$, and where the symmetric elastic deformation tensor

$$\bar{\mathbf{C}} = \bar{C}_{\alpha\beta}\tilde{\mathbf{g}}^{\alpha} \otimes \tilde{\mathbf{g}}^{\beta} = \bar{F}^{a}_{\cdot\alpha} g_{ab} \bar{F}^{b}_{\cdot\beta} \tilde{\mathbf{g}}^{\alpha} \otimes \tilde{\mathbf{g}}^{\beta}, \qquad \bar{C}_{\alpha\beta} = \bar{F}^{a}_{\cdot\alpha} g_{ab} \bar{F}^{b}_{\cdot\beta} = \bar{C}_{\beta\alpha}. \tag{271}$$

The dependence of strain energy on other terms (e.g. internal state variables representing contributions from lattice defects, or X representing heterogeneous material properties) can be incorporated without severely affecting subsequent arguments. Furthermore, strain energy per unit reference volume $\tilde{J}\,\Psi$ could be used alternatively without conceptual difficulties; in the usual case of isochoric plastic deformation from slip, the distinction is irrelevant because $\tilde{J} = 1$. For a hyperelastic response, the stress obeys

$$P_{a}^{\cdot\alpha} = g_{ab}P^{b\alpha} = \frac{\partial \Psi}{\partial \bar{F}^{a}_{\cdot\alpha}} = \frac{\partial \Psi}{\partial \bar{C}_{\beta\chi}} \frac{\partial \bar{C}_{\beta\chi}}{\partial \bar{F}^{a}_{\cdot\alpha}} = 2 g_{ab} \frac{\partial \Psi}{\partial \bar{C}_{\alpha\beta}} \bar{F}^{b}_{\cdot\beta}, \tag{272}$$

where the chain rule and the following identity have been used:

$$\frac{\partial \bar{C}_{\alpha\beta}}{\partial \bar{F}^{a}_{\cdot\chi}} = 2 g_{ab} \delta^{\chi}_{(\alpha} \bar{F}^{b}_{\cdot\beta)}. \tag{273}$$

Constitutive equation (272), in conjunction with definition (268), is standard for crystalline solids and can be derived using thermodynamic arguments [14, 19, 25, 33]. Functional forms of Ψ for nonlinear elasticity of anisotropic single crystals belonging to various crystal classes are available in the literature [14, 32]. Usually, dependence on $\tilde{\mathbf{g}}$ is not written explicitly; however, as will be demonstrated shortly, inverse metric components $\tilde{g}^{\alpha\beta}$ are needed for construction of scalar invariants of covariant deformation tensor $\bar{C}_{\alpha\beta}$ entering the strain energy function, e.g. the trace $\mathrm{tr}\bar{\mathbf{C}} = \bar{C}^{\alpha}_{\cdot\alpha} = \bar{C}_{\alpha\beta}\tilde{g}^{\alpha\beta}$ and Jacobian determinant $\bar{J} = \det(\bar{F}^{a}_{\cdot\alpha})[\det(g_{ab})\det(\tilde{g}^{\alpha\beta})]^{1/2} = [\det(\bar{C}^{\alpha}_{\cdot\beta})]^{1/2}$.

For illustrative purposes, consider an isotropic (poly)crystalline solid whose nonlinear hyperelastic response can be described via the neo-Hookean strain energy potential [46]:

$$\Psi = \frac{\lambda}{2}[\mathrm{tr}\bar{\mathbf{C}} + (\ln\bar{J})^2 - 3] - \mu \ln\bar{J} = \frac{\lambda}{2}(\bar{C}_{\alpha\beta}\tilde{g}^{\alpha\beta} - 3) + \ln\bar{J}\left(\frac{\lambda}{2}\ln\bar{J} - \mu\right), \tag{274}$$

where λ and μ are elastic constants. Using (253), the stress of (272) for a neo-Hookean material described by (274) is

$$P^{a\alpha} = \mu(\bar{F}^{a\alpha} - \bar{F}^{-1\alpha a}) + \lambda\bar{F}^{-1\alpha a}\ln\bar{J} = \mu(\bar{F}^{a}_{\cdot\beta}\tilde{g}^{\alpha\beta} - \bar{F}^{-1\alpha}_{\cdot b}g^{ab}) + \lambda\bar{F}^{-1\alpha}_{\cdot b}g^{ab}\ln\bar{J}. \tag{275}$$

The partial derivative of the stress entering (269) is then

$$\partial_{\alpha}P^{a\alpha} = \bar{F}^{b}_{\cdot\alpha}\partial_{b}P^{a\alpha} = \mu\bar{F}^{b}_{\cdot\alpha}\partial_{b}\bar{F}^{a\alpha} - (\mu - \lambda\ln\bar{J})\bar{F}^{b}_{\cdot\alpha}\partial_{b}\bar{F}^{-1\alpha a} + \lambda\bar{J}^{-1}g^{ab}\partial_{b}\bar{J}. \tag{276}$$

The second and third terms on the right-hand side of (276) can be written in terms of the partial derivative of the elastic deformation gradient rather than its inverse and determinant upon use of the chain rule and the identities

$$\frac{\partial \bar{F}^{-1\alpha}_{\cdot a}}{\partial \bar{F}^{b}_{\cdot\beta}} = -\bar{F}^{-1\alpha}_{\cdot b}\bar{F}^{-1\beta}_{\cdot a}, \quad \partial_{a}\bar{J} = \frac{\partial\bar{J}}{\partial\bar{F}^{b}_{\cdot\beta}}\partial_{a}\bar{F}^{b}_{\cdot\beta} + \frac{\partial\bar{J}}{\partial(g/\tilde{g})}\partial_{a}(g/\tilde{g}) = \bar{J}\bar{F}^{-1\beta}_{\cdot b}\partial_{a}\bar{F}^{b}_{\cdot\beta} + \bar{J}\,\partial_{a}[\ln(g/\tilde{g})], \tag{277}$$

leading to

$$\partial_{\alpha}P^{a\alpha} = \mu\bar{F}^{b}_{\cdot\alpha}\partial_{b}\bar{F}^{a\alpha} + (\mu - \lambda\ln\bar{J})(\bar{F}^{-1\cdot a}_{\alpha}\partial_{b}\bar{F}^{b\alpha} + \bar{F}^{-1\alpha a}\bar{F}^{b\beta}\partial_{b}\tilde{g}_{\alpha\beta} - \partial_{b}g^{ab}) + \lambda g^{ab}\{\bar{F}^{-1\alpha}_{\cdot c}\partial_{b}\bar{F}^{c}_{\cdot\alpha} + \partial_{b}[\ln(g/\tilde{g})]\}. \tag{278}$$

With stress components computed generically using (272) or specifically (e.g. (275)), all that remains in (269) is specification of intermediate connection coefficients $\Gamma^{\cdot\cdot\alpha}_{\beta\chi}$, which can be defined as discussed in Section 3.2.2. Explicit forms are considered in what follows.

4.2.1. Holonomic elastic deformation. When (108) applies, such that $\tilde{B}$ can be regarded as a Euclidean space, then (190) can be used to define the intermediate connection coefficients. The right-most term of (269) vanishes from the integrability of $\bar{F}^{-1\alpha}_{\ \ \cdot a} = \partial_a \tilde{x}^\alpha$ and the symmetry of $\Gamma^{\cdot\cdot\alpha}_{\beta\chi}$ in (190), leaving

$$\partial_\alpha P^{a\alpha} + \Gamma^{\cdot\cdot\alpha}_{\alpha\beta} P^{a\beta} + \overset{g}{\Gamma}{}^{\cdot\cdot a}_{bc} \bar{F}^b_{\cdot\alpha} P^{c\alpha} = 0. \tag{279}$$

4.2.2. Coincident referential and intermediate coordinate frames. Next consider the case when (109) may apply, such that single-valued coordinates $\tilde{x}^\alpha(X,t)$ continuously differentiable with respect to X^A may not exist in $\tilde{B}$. Let identical coordinate systems and metric tensors be assigned at to each material particle X in configurations B_0 and $\tilde{B}$ [36], such that the appropriate connection coefficients are given by (196). In this case, linear momentum balance (269) becomes

$$\begin{aligned}\partial_\alpha P^{a\alpha} &+ \delta^\alpha_A \delta^{CG}_\beta \overset{}{\Gamma}{}^{\cdot\cdot A}_{BC} \tilde{F}^{-1B}_{\ \ \cdot\alpha} P^{a\beta} + \overset{g}{\Gamma}{}^{\cdot\cdot a}_{bc} \bar{F}^b_{\cdot\alpha} P^{c\alpha} \\ &+ (\det \bar{\mathbf{F}}) e_{\alpha\beta\chi} e^{dbc} P^{a\alpha} \bar{F}^{-1\beta}_{\ \ \cdot d} (\partial_c \bar{F}^{-1\chi}_{\ \ \cdot b} + \delta^\chi_A \delta^{CG}_\epsilon \overset{}{\Gamma}{}^{\cdot\cdot A}_{BC} F^{-1B}_{\ \ \cdot c} \bar{F}^{-1\epsilon}_{\ \ \cdot b}) = 0.\end{aligned} \tag{280}$$

4.2.3. Coincident spatial and intermediate coordinate frames. Now consider the case when (113) may apply, such that single-valued coordinates $\tilde{x}^\alpha(x,t)$ continuously differentiable with respect to x^a may not exist in $\tilde{B}$. Identical coordinate systems and metric tensors are assigned to each spatial point x in configurations B and $\tilde{B}$, so that appropriate connection coefficients on $\tilde{B}$ are given by (206) (see also the Appendix for a specific example of cylindrical coordinates). Linear momentum balance (269) becomes

$$\begin{aligned}\partial_\alpha P^{a\alpha} &+ \overset{g}{\Gamma}{}^{\cdot\cdot a}_{bc} \bar{F}^b_{\cdot\alpha} (\delta^\alpha_a \delta^c_\beta P^{a\beta} + P^{c\alpha}) \\ &+ (\det \bar{\mathbf{F}}) e_{\alpha\beta\chi} e^{dbc} P^{a\alpha} \bar{F}^{-1\beta}_{\ \ \cdot d} (\partial_c \bar{F}^{-1\chi}_{\ \ \cdot b} + \delta^\chi_e \delta^{hg}_\epsilon \overset{g}{\Gamma}{}^{\cdot\cdot e}_{ch} \bar{F}^{-1\epsilon}_{\ \ \cdot b}) = 0.\end{aligned} \tag{281}$$

Choice (206) appears favorable over (196) in the context of the balance of linear momentum because (281) involves only one set of connection coefficients and one deformation map ($\bar{\mathbf{F}}$), in contrast to (280) which involves two different sets of connection coefficients and both deformation maps $\bar{\mathbf{F}}$ and $\tilde{\mathbf{F}}$. When (278) applies, for example, momentum balance (281) can be expressed completely in terms of the elastic constants, elastic deformation $\bar{\mathbf{F}}$ (including its gradient, inverse, and determinant), spatial metric $\mathbf{g}$, and spatial connection coefficients $\overset{g}{\Gamma}{}^{\cdot\cdot a}_{bc}$, the latter which are obtained from partial derivatives of $\mathbf{g}$ through (53).

4.2.4. Cartesian intermediate coordinates. Finally consider the case wherein Cartesian basis vectors are used on $\tilde{B}$, so that (214) holds meaning $\Gamma^{\cdot\cdot\alpha}_{\beta\chi} = 0$. The linear momentum balance (269) reduces to

$$\partial_\alpha P^{a\alpha} + \overset{g}{\Gamma}{}^{\cdot\cdot a}_{bc} \bar{F}^b_{\cdot\alpha} P^{c\alpha} + (\det \bar{\mathbf{F}}) e_{\alpha\beta\chi} e^{dbc} P^{a\alpha} \bar{F}^{-1\beta}_{\ \ \cdot d} \partial_c \bar{F}^{-1\chi}_{\ \ \cdot b} = 0. \tag{282}$$

When Cartesian spatial coordinates are used and when $\bar{\mathbf{F}}^{-1}$ is integrable as in (279), then (282) reduces to $\partial_\alpha P^{a\alpha} = 0$.

5. Conclusions

The possibly anholonomic intermediate configuration of a deformable body has been examined from the perspectives of tensor calculus and differential geometry. Various choices of extrinsic coordinate systems for the intermediate configuration have been considered; corresponding metric tensors, connection coefficients, torsion, and curvature have been derived. Partial and total covariant differentiation with respect to anholonomic coordinates have been defined. It has been shown that when the same curvilinear coordinate systems are prescribed on intermediate and Euclidean reference or current configurations, the torsion and curvature of the connection coefficients in the intermediate configuration, which are generally time dependent, need not vanish. Conventional identities that hold in reference and current configurations such as vanishing divergence of the curl of a vector field and vanishing curl of the gradient of a scalar field do not necessarily apply in the intermediate configuration. The balance of linear momentum has been derived for a hyperelastic–plastic material in general curvilinear coordinates. For neo-Hookean elasticity, the choice of coincident spatial and intermediate coordinate systems enables the stress divergence to be expressed completely in terms of elastic constants, spatial gradients of elastic deformation, and the spatial metric tensor and its spatial derivatives.

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Conflict of interest

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References

[1] Eckart, C. The thermodynamics of irreversible processes. IV. The theory of elasticity and anelasticity. *Phys Rev* 1948; 73: 373–382.
[2] Bilby, BA, Gardner, LRT, and Stroh, AN. Continuous distributions of dislocations and the theory of plasticity. In *Proceedings of the 9th International Congress of Applied Mechanics*, vol. 8. Brussels: Université de Bruxelles, 1957, pp. 35–44.
[3] Kröner, E. Allgemeine kontinuumstheorie der versetzungen und eigenspannungen. *Arch Rat Mech Anal*, 4:273–334, 1960.
[4] Bilby, BA, Bullough, R, and Smith, E. Continuous distributions of dislocations: a new application of the methods of non-Riemannian geometry. *Proc R Soc A* 1955; 231: 263–273.
[5] Kondo, K. On the analytical and physical foundations of the theory of dislocations and yielding by the differential geometry of continua. *Int J Eng Sci* 1964; 2: 219–251.
[6] Noll, W. Materially uniform simple bodies with inhomogeneities. *Arch Rat Mech Anal* 1967; 27: 1–32.
[7] Wang, C-C. On the geometric structures of simple bodies, a mathematical foundation for the theory of continuous distributions of dislocations. *Arch Rat Mech Anal* 1967; 27: 33–94.
[8] Cermelli, P, and Sellers, S. On the nonlinear mechanics of Bravais crystals with continuous distributions of defects. *Math Mech Solids* 1998; 3: 331–358.
[9] Acharya, A, and Bassani, JL. Lattice incompatibility and a gradient theory of crystal plasticity. *J Mech Phys Solids* 2000; 48: 1565–1595.
[10] Clayton, JD, Bammann, DJ, and McDowell, DL. A geometric framework for the kinematics of crystals with defects. *Philos Mag* 2005; 85: 3983–4010.
[11] Gupta, A, Steigmann, DJ, and Stolken, JS. On the evolution of plasticity and incompatibility. *Math Mech Solids* 2007; 12: 583–610.
[12] Cleja-Tigoiu, S, Fortune, D, and Vallee, C. Torsion equation in anisotropic elasto-plastic materials with continuously distributed dislocations. *Math Mech Solids* 2008; 13: 667–689.
[13] Grammenoudis, P, and Tsakmakis, C. Plastic intermediate configuration and related spatial differential operators in micromorphic plasticity. *Math Mech Solids* 2010; 15: 515–538.
[14] Clayton, JD. *Nonlinear Mechanics of Crystals*. Dordrecht: Springer, 2011.
[15] Stojanovitch, R. On the stress relation in non-linear thermoelasticity. *Int J Non-Lin Mech* 1969; 4: 217–233.
[16] Rodriguez, EK, Hoger, A, and McCulloch, AD. Stress-dependent finite growth in soft elastic tissues. *J Biomech* 1994; 27: 455–467.
[17] Clayton, JD. A non-linear model for elastic dielectric crystals with mobile vacancies. *Int J Non-Lin Mech* 2009; 44: 675–688.
[18] Bammann, DJ, and Solanki, KN. On kinematic, thermodynamic, and kinetic coupling of a damage theory for polycrystalline material. *Int J Plasticity* 2010; 26: 775–793.
[19] Clayton, JD. A continuum description of nonlinear elasticity, slip and twinning, with application to sapphire. *Proc R Soc A* 2009; 465: 307–334.
[20] Clayton, JD, and Knap, J. A phase field model of deformation twinning: nonlinear theory and numerical simulations. *Phys D Nonlin Phenomena* 2011; 240: 841–858.
[21] Lardner, RW. Foundations of the theory of disclinations. *Arch Mech* 1973; 25: 911–922.
[22] Clayton, JD, McDowell, DL, and Bammann, DJ. Modeling dislocations and disclinations with finite micropolar elastoplasticity. *Int J Plasticity* 2006; 22: 210–256.
[23] Wang, C-C, and Truesdell, C. *Introduction to Rational Elasticity*. Leyden: Noordhoff, 1973.

[24] Clayton, JD, and McDowell, DL. A multiscale multiplicative decomposition for elastoplasticity of polycrystals. *Int J Plasticity* 2003; 19: 1401–1444.
[25] Lubarda, VA. Constitutive theories based on the multiplicative decomposition of deformation gradient: thermoelasticity, elastoplasticity, and biomechanics. *Appl Mech Rev* 2004; 57: 95–108.
[26] Yavari, A. A geometric theory of growth mechanics. *J Nonlin Sci* 2010; 20: 781–830.
[27] Schouten, JA. *Ricci Calculus*. Berlin: Springer-Verlag, 1954.
[28] Clayton, JD, Bammann, DJ, and McDowell, DL. Anholonomic configuration spaces and metric tensors in finite elastoplasticity. *Int J Non-Lin Mech* 2004; 39: 1039–1049.
[29] Vranceanu, G. Sur les espaces non holonomes. *C R Acad Sci* 1926; 183: 852–854.
[30] Ericksen, JL. Tensor fields. In Flugge S (ed.), *Handbuch der Physik*, vol. III/1. Berlin: Springer-Verlag, 1960, pp. 794–858.
[31] Kondo, K. Fundamentals of the theory of yielding elementary and more intrinsic expositions: Riemannian and non-Riemannian terminology. *Matrix Tensor Q* 1984; 34: 55–63.
[32] Teodosiu, C. *Elastic Models of Crystal Defects*. Berlin: Springer-Verlag, 1982.
[33] Havner, KS. *Finite Plastic Deformation of Crystalline Solids*. Cambridge: Cambridge University Press, 1993.
[34] Teodosiu, C. Contributions to the continuum theory of dislocations and initial stresses. III. *Rev Roumaine Sci Tech Mec Appl* 1967; 12: 1291–1308.
[35] Kaluza, M, and Le, KC. On torsion of a single crystal rod. *Int J Plasticity* 2011; 27: 460–469.
[36] Simo, JC. A framework for finite strain elastoplasticity based on maximum plastic dissipation and the multiplicative decomposition: Part I. Continuum formulation. *Comput Meth Appl Mech Eng* 1988; 66: 199–219.
[37] Truesdell, CA, and Toupin, RA. The classical field theories. In Flugge S (ed.), *Handbuch der Physik*, vol. III/1. Berlin: Springer-Verlag, 1960, pp. 226–793.
[38] Marsden, JE, and Hughes, TJR. *Mathematical Theory of Elasticity*. Englewood Cliffs, NJ: Prentice-Hall, 1983.
[39] Eringen, AC. *Nonlinear Theory of Continuous Media*. New York: McGraw-Hill, 1962.
[40] Menzel, A, and Steinmann, P. On configurational forces in multiplicative elastoplasticity. *Int J Solids Structures* 2007; 44: 4442–4471.
[41] Toupin, RA. Theories of elasticity with couple stress. *Arch Rat Mech Anal* 1964; 17: 85–112.
[42] Steinmann, P. Views on multiplicative elastoplasticity and the continuum theory of dislocations. *Int J Eng Sci* 1996; 34: 1717–1735.
[43] Gurtin, ME. A gradient theory of single-crystal viscoplasticity that accounts for geometrically necessary dislocations. *J Mech Phys Solids* 2002; 50: 5–32.
[44] Edelen, DGB, and Lagoudas, DC. *Gauge Theory and Defects in Solids*. Amsterdam: North-Holland, 1988.
[45] Clayton, JD, McDowell, DL, and Bammann, DJ. A multiscale gradient theory for single crystalline elastoviscoplasticity. *Int J Eng Sci* 2004; 42: 427–457.
[46] Clayton, JD, and Knap, J. Phase field modeling of twinning in indentation of transparent crystals. *Modell Simul Mater Sci Eng* 2011; 19: 085005.

Appendix: Anholonomic cylindrical coordinates

As an illustrative example of curvilinear intermediate (anholonomic) coordinates, consider cylindrical spatial coordinates mapped to the intermediate configuration. Because cylindrical spatial coordinates exhibit relatively simple forms for metric tensors and connection coefficients, corresponding quantities mapped to the intermediate configuration can be derived by inspection.

In this example, (146)–(151) apply for basis vectors and metric tensors. Since the angular coordinate in current configuration B is labeled θ, overbars are used here to denote particular (anholonomic) coordinates referred to the intermediate configuration $\tilde{B}$. Specifically, holonomic coordinates on B are denoted by the usual (r, θ, z), while anholonomic coordinates on $\tilde{B}$ are denoted by $(\bar{r}, \bar{\theta}, \bar{z})$. In what follows, indices (r, θ, z) and $(\bar{r}, \bar{\theta}, \bar{z})$ refer to specific coordinates and are exempt from the summation convention. Generic indices $(a, b, c, d \ldots)$ and $(\alpha, \beta, \chi, \delta \ldots)$ refer to free spatial and intermediate quantities, respectively, and are subject to summation over repeated indices.

A.1. Spatial components

Quantities associated with the usual holonomic spatial coordinates are considered first [39]. In three spatial dimensions, cylindrical coordinates are

$$(x^1, x^2, x^3) \to (r, \theta, z), \qquad r \geq 0, \qquad \theta \in (-\pi, \pi]. \tag{283}$$

The squared length of a differential line element $d\mathbf{x}$ is

$$d\mathbf{x} \cdot d\mathbf{x} = (dr)^2 + (r\,d\theta)^2 + (dz)^2. \tag{284}$$

The metric tensor and its inverse are

$$[g_{ab}] = \begin{bmatrix} g_{rr} & g_{r\theta} & g_{rz} \\ g_{\theta r} & g_{\theta\theta} & g_{\theta z} \\ g_{zr} & g_{z\theta} & g_{zz} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [g^{ab}] = \begin{bmatrix} g^{rr} & g^{r\theta} & g^{rz} \\ g^{\theta r} & g^{\theta\theta} & g^{\theta z} \\ g^{zr} & g^{z\theta} & g^{zz} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{285}$$

Determinants of the metric tensor and its inverse are

$$g = \det(g_{ab}) = r^2, \qquad g^{-1} = \det(g^{ab}) = 1/r^2. \tag{286}$$

A differential volume element is

$$dv = \sqrt{g}\, dx^1\, dx^2\, dx^3 = r\, dr\, d\theta\, dz. \tag{287}$$

Connection coefficients from (53) are

$$\overset{g}{\Gamma}{}_{r\theta}^{\cdot\cdot\theta} = \overset{g}{\Gamma}{}_{\theta r}^{\cdot\cdot\theta} = 1/r, \quad \overset{g}{\Gamma}{}_{\theta\theta}^{\cdot\cdot r} = -r, \quad \overset{g}{\Gamma}{}_{bc}^{\cdot\cdot a} = 0 \text{ otherwise}. \tag{288}$$

It is remarked that connection coefficients (288) are symmetric in covariant indices (i.e. null torsion), and that the Riemann–Christoffel curvature tensor formed from these coefficients vanishes. Let $\mathbf{x}$ denote the position vector in Euclidean space. Natural basis vectors are

$$\mathbf{g}_r = \partial_r \mathbf{x}, \quad \mathbf{g}_\theta = \partial_\theta \mathbf{x}, \quad \mathbf{g}_z = \partial_z \mathbf{x}. \tag{289}$$

A.2. Intermediate components

Intermediate basis vectors, from (289) and (146), are

$$\tilde{\mathbf{g}}_{\bar{r}}(x) = \partial_r \mathbf{x}, \quad \tilde{\mathbf{g}}_{\bar{\theta}}(x) = \partial_\theta \mathbf{x}, \quad \tilde{\mathbf{g}}_{\bar{z}}(x) = \partial_z \mathbf{x}. \tag{290}$$

The metric tensor and its inverse, from (285) and (147), are

$$[\tilde{g}_{\alpha\beta}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\tilde{g}^{\alpha\beta}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{291}$$

Determinants of the intermediate metric tensor and its inverse, from (286) and (149), are

$$\tilde{g} = r^2, \quad \tilde{g}^{-1} = 1/r^2. \tag{292}$$

To second order in $d\mathbf{x}$, from (226), the squared length of an intermediate line element is

$$d\tilde{\mathbf{x}} \cdot d\tilde{\mathbf{x}} = (\bar{F}^{-1\alpha}_{\ \ \cdot r}\, dr + \bar{F}^{-1\alpha}_{\ \ \cdot\theta}\, d\theta + \bar{F}^{-1\alpha}_{\ \ \cdot z}\, dz)\tilde{g}_{\alpha\beta}(\bar{F}^{-1\beta}_{\ \ \cdot r}\, dr + \bar{F}^{-1\beta}_{\ \ \cdot\theta}\, d\theta + \bar{F}^{-1\beta}_{\ \ \cdot z}\, dz). \tag{293}$$

An intermediate volume element is, from (151), (249), (251), and (287),

$$d\tilde{V} = (\det \bar{\mathbf{F}}^{-1}) r\, dr\, d\theta\, dz. \tag{294}$$

In the present example, definitions (206), (208), and (209) apply for connection coefficients, torsion, and curvature referred to the intermediate configuration. From (288) and (206), anholonomic connection coefficients consist of up to nine non-zero components:

$$\Gamma_{\bar{r}\bar{\theta}}^{\cdot\cdot\bar{\theta}} = \bar{F}^r_{\bar{r}}/r, \quad \Gamma_{\bar{\theta}\bar{\theta}}^{\cdot\cdot\bar{\theta}} = \bar{F}^r_{\bar{\theta}}/r, \quad \Gamma_{\bar{z}\bar{\theta}}^{\cdot\cdot\bar{\theta}} = \bar{F}^r_{\bar{z}}/r; \tag{295}$$

$$\Gamma_{\bar{r}\bar{r}}^{\cdot\cdot\bar{\theta}} = \bar{F}^{\theta}_{\bar{r}}/r, \quad \Gamma_{\bar{\theta}\bar{r}}^{\cdot\cdot\bar{\theta}} = \bar{F}^{\theta}_{\bar{\theta}}/r, \quad \Gamma_{\bar{z}\bar{r}}^{\cdot\cdot\bar{\theta}} = \bar{F}^{\theta}_{\bar{z}}/r; \tag{296}$$

$$\Gamma_{\bar{r}\bar{\theta}}^{\cdot\cdot\bar{r}} = -\bar{F}^{\theta}_{\bar{r}}r, \quad \Gamma_{\bar{\theta}\bar{\theta}}^{\cdot\cdot\bar{r}} = -\bar{F}^{\theta}_{\bar{\theta}}r, \quad \Gamma_{\bar{z}\bar{\theta}}^{\cdot\cdot\bar{r}} = -\bar{F}^{\theta}_{\bar{z}}r. \tag{297}$$

Possibly non-zero skew covariant components of these coefficients are

$$\Gamma_{[\bar{r}\bar{\theta}]}^{\cdot\cdot\bar{\theta}} = \frac{1}{2r}(\bar{F}^{r}_{\bar{r}} - \bar{F}^{\theta}_{\bar{\theta}}), \quad \Gamma_{[\bar{z}\bar{\theta}]}^{\cdot\cdot\bar{\theta}} = \frac{1}{2r}\bar{F}^{r}_{\bar{z}}; \tag{298}$$

$$\Gamma_{[\bar{z}\bar{r}]}^{\cdot\cdot\bar{\theta}} = \frac{1}{2r}\bar{F}^{\theta}_{\bar{z}}; \tag{299}$$

$$\Gamma_{[\bar{r}\bar{\theta}]}^{\cdot\cdot\bar{r}} = -\frac{r}{2}\bar{F}^{\theta}_{\bar{r}}, \quad \Gamma_{[\bar{z}\bar{\theta}]}^{\cdot\cdot\bar{r}} = -\frac{r}{2}\bar{F}^{\theta}_{\bar{z}}. \tag{300}$$

Torsion can then be computed from (208), noting that the anholonomic object

$$\bar{\kappa}_{\beta\chi}^{\cdot\cdot\alpha} = \bar{F}^{a}_{\cdot\beta}\bar{F}^{b}_{\cdot\chi}\,\partial_{[a}\bar{F}^{-1\alpha}{}_{\cdot b]}. \tag{301}$$

Spatial connection coefficients (288) have the following non-zero skew partial derivatives:

$$\partial_{[r}\overset{g}{\Gamma}{}^{\cdot\cdot\theta}_{\theta]r} = -\frac{1}{2r^2}, \quad \partial_{[r}\overset{g}{\Gamma}{}^{\cdot\cdot r}_{\theta]\theta} = -\frac{1}{2}. \tag{302}$$

Therefore, possibly non-vanishing components of the anholonomic curvature (209) follow as

$$R_{\beta\chi\bar{r}}^{\cdot\cdot\cdot\bar{\theta}} = \frac{2}{r^2}\bar{F}^{r}_{\cdot[\beta}\bar{F}^{\theta}_{\cdot\chi]}, \quad R_{\beta\chi\bar{\theta}}^{\cdot\cdot\cdot\bar{r}} = 2\bar{F}^{r}_{\cdot[\beta}\bar{F}^{\theta}_{\cdot\chi]}. \tag{303}$$

Written out completely with no free indices, (303) yields the components

$$R_{\bar{r}\bar{\theta}\bar{r}}^{\cdot\cdot\cdot\bar{\theta}} = \frac{1}{r^2}(\bar{F}^{r}_{\cdot\bar{r}}\bar{F}^{\theta}_{\cdot\bar{\theta}} - \bar{F}^{r}_{\cdot\bar{\theta}}\bar{F}^{\theta}_{\cdot\bar{r}}), \quad R_{\bar{\theta}\bar{z}\bar{r}}^{\cdot\cdot\cdot\bar{\theta}} = \frac{1}{r^2}(\bar{F}^{r}_{\cdot\bar{\theta}}\bar{F}^{\theta}_{\cdot\bar{z}} - \bar{F}^{r}_{\cdot\bar{z}}\bar{F}^{\theta}_{\cdot\bar{\theta}}), \quad R_{\bar{z}\bar{r}\bar{r}}^{\cdot\cdot\cdot\bar{\theta}} = \frac{1}{r^2}(\bar{F}^{r}_{\cdot\bar{z}}\bar{F}^{\theta}_{\cdot\bar{r}} - \bar{F}^{r}_{\cdot\bar{r}}\bar{F}^{\theta}_{\cdot\bar{z}}); \tag{304}$$

$$R_{\bar{r}\bar{\theta}\bar{\theta}}^{\cdot\cdot\cdot\bar{r}} = \bar{F}^{r}_{\cdot\bar{r}}\bar{F}^{\theta}_{\cdot\bar{\theta}} - \bar{F}^{r}_{\cdot\bar{\theta}}\bar{F}^{\theta}_{\cdot\bar{r}}, \quad R_{\bar{\theta}\bar{z}\bar{\theta}}^{\cdot\cdot\cdot\bar{r}} = \bar{F}^{r}_{\cdot\bar{\theta}}\bar{F}^{\theta}_{\cdot\bar{z}} - \bar{F}^{r}_{\cdot\bar{z}}\bar{F}^{\theta}_{\cdot\bar{\theta}}, \quad R_{\bar{z}\bar{r}\bar{\theta}}^{\cdot\cdot\cdot\bar{r}} = \bar{F}^{r}_{\cdot\bar{z}}\bar{F}^{\theta}_{\cdot\bar{r}} - \bar{F}^{r}_{\cdot\bar{r}}\bar{F}^{\theta}_{\cdot\bar{z}}. \tag{305}$$

Clearly, skew components of the connection (and, hence, the torsion) as well as the curvature need not vanish even if the anholonomic object of (301) is identically zero.

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